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THE "LINEAR" LIMIT OF THIN FILM FLOWS AS AN OBSTACLE-TYPE FREE BOUNDARY PROBLEM*

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Abstract. We study the limit as $n \to 0$ of the nonnegative, self-similar source-type solutions of the *thin film equation* $u_t + (u^n u_{xxx})_x = 0$. We obtain a unique limiting function u, which is a solution of an *obstacle-type free boundary problem*, with the constraint $u \geq 0$, associated with the *linear equation* $u_t + u_{xxxx} = 0$. The function u is C^1 for $t > 0$ and has finite speed of propagation, the positivity set $\{u > 0\}$ being bounded by two *contact lines* $x = \pm a t^{1/4}$ (a constant). The function u has a Dirac mass as initial condition and satisfies the linear equation in the positivity set, but not across the *free boundaries* or contact lines, also known as *moving boundaries*. We give an integral representation of u in the positivity set. We set up a precise definition of the general (non-self-similar) obstacle-type free boundary problem, which is different from a standard parabolic variational inequality, and compare it with the Cauchy problem. We also consider source-type solutions for negative values of n, which are solutions of the obstacle-type free boundary problem (rather than the Cauchy problem) and still have finite speed of propagation.

The situation is rather different from that of the heat equation $u_t = u_{xx}$ and the *porous media equation* $u_t = (u^n u_x)_x$ in the *fast diffusion range* $n < 0$. For these second-order equations the current problems have globally positive solutions. Hence, they have infinite speed of propagation and the condition $u \geq 0$ does not generate obstacle problems.

Key words. thin films, higher-order diffusion, nonlinear diffusion, obstacle problem, free boundary, moving boundary, self-similarity, source-type solutions

AMS subject classifications. 35K65, 35R35, 35K85, 76D08

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1. Introduction.

A first objective of this paper is to study the limit as $n \to 0^+$ of the nonnegative, self-similar source-type solutions of the *thin film equation*

$$u_t + (u^n u_{xxx})_x = 0\,. \tag{1.1}$$

We obtain a unique limiting function, which is a solution of an *obstacle-type free boundary problem*, with the constraint $u \geq 0$, associated with the *linear equation*

$$u_t + u_{xxxx} = 0\,. \tag{1.2}$$

The source solutions for $n > 0$ were investigated in [11].

The linear equation (1.2) belongs to a large class of fourth-order (linear and nonlinear) equations which do *not* preserve the sign of the initial data [5], [38]. This is in sharp contrast to the properties of the heat equation and other second-order parabolic equations. During the last 10 years it has been established that the thin

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film equation for $n > 0$, unlike (1.2), preserves the nonnegativity of the initial data [9], [4], [14]. Hence the limit as $n \to 0^+$ of the Cauchy problem for (1.1) cannot be the Cauchy problem for the linear equation (1.2). This limiting problem has to include the condition $u \geq 0$ as a constraint and in this sense is an *obstacle problem*. Furthermore, the boundary of the domain $\{u > 0\}$ is not known in advance but has to be determined as part of the solution: we have a *moving boundary problem* or *free boundary problem*. A part of the literature uses the term “free boundary problem” when the boundary is stationary and “moving boundary problem” when the boundary is time dependent, but we shall consider both terms as interchangeable. Although the physical formulation of moving boundary problems is in many cases very transparent, the analytical study of them presents many difficulties. This motivated the introduction of a variational reformulation of moving boundary problems, giving rise to the *variational inequalities* [42], where the domain is fixed and the study of the free boundary is postponed and separated from the formulation of the problem. We refer to the books [32], [26], and [37] for background material on free boundary problems and the connection between applied problems and variational inequalities.

Then two natural questions arise: (1) How does the obstacle-type free boundary problem for (1.1) compare with the Cauchy problem when $n \geq 0$? And (2) Is the limit as $n \to 0^+$ of the Cauchy problem for (1.1) a standard parabolic variational inequality? In order to address these questions we set up a precise definition of the general (non-self-similar) obstacle-type free boundary problem. The definition is worked out in the frame of an expository overview of the current theory for the thin film equation in one space dimension. We discuss the first question in two stages: a sketch is outlined in this introduction and a more detailed study is given in section 2. At the end of section 2 we give reasons indicating that the answer to the second question is negative.

Once the general frame is presented, we come back to the self-similar case. The obstacle-type free boundary problem indicates a natural way to introduce the concept of source-type solution for $n \leq 0$. In this range of exponents we obtain results on existence, uniqueness, and finite speed of propagation for source solutions (Theorem A). The limiting behavior as $n \to 0^+$ is stated in Theorem B. Finally, the methods used to study this limiting process are applied to derive explicit estimates and properties of continuity with respect to the parameter n in the range $-4 < n < 3$. The estimates may provide useful tests for numerical research.

Equation (1.1) arises in lubrication models for thin viscous films, spreading droplets and Hele–Shaw cells driven by surface tension. In the last few years the full range $0 < n \leq 3$ has been considered in the literature from a modeling point of view; see the survey papers [13], [6], [43], [44]. The works [3] and [15] contain new modeling approaches. In [46] a singular perturbation expansion about $n = 0$ is performed. The very recent paper [41] carries out comprehensive research by means of matched asymptotics and singular perturbations, including negative values of n. (See also [22] for bounded domains.) Another recent work [21] constructs numerically the source solutions for negative n. Negative exponents are also considered in [12] for a fourth-order equation close to (1.1), namely, $u_t + |u|^n u_{xxxx} = 0$. Although the present paper deals only with the one-dimensional case, we refer to [16], [28], [27], and references therein for a rigorous analytical study of the thin film equation in higher dimensions.

For $n > 0$ a source-type solution of the thin film equation is a “strong” solution (see section 2) of the Cauchy problem for (1.1) satisfying the initial condition

$$u(\cdot, t) \to M\,\delta \quad \text{as } t \to 0^+ ,$$

where δ stands for the Dirac mass and M is a given constant. We assume $M > 0$.

This initial condition means that

$$\int_{\mathbb{R}} u(x,t)\varphi(x)\,dx \to M\varphi(0) \quad \text{as } t \to 0^+ \quad \text{for all } \varphi \in C_c(\mathbb{R}),$$

where the subscript c stands for compact support.

As already noted, nonnegative, self-similar source-type solutions were considered in [11] for one space dimension. (See [34] for higher dimensions.) These papers prove theorems on existence and uniqueness if $0 < n < 3$ and on nonexistence if $n \geq 3$.

Let us proceed with some remarks on the general theory for the thin film equation. We present here a sketch that will be made precise in section 2. Given an initial datum $u_0 \geq 0$ with compact support, the Cauchy problem for (1.1) has a *nonnegative* strong solution u with *finite speed of propagation* if $0 < n < 3$; see [7], [40], [27], and references therein. The components of the boundary of the set $\{u > 0\}$, excluding the x-axis, are known as *free boundaries*, *interfaces*, or *contact lines*. Strong solutions can be described (up to some technical details) as classical solutions in the positivity set $\{u > 0\}$ that enjoy the *mass conservation property*

$$\int_{\mathbb{R}} u(x,t)\,dx = M \quad \text{for all } t > 0 \tag{1.3}$$

and satisfy the conditions

$$u = u_x = u^n u_{xxx} = 0 \tag{1.4}$$

at the free boundaries. Hence these solutions satisfy the *zero contact angle* condition $u_x = 0$ and the *zero flux* condition $u^n u_{xxx} = 0$ at the interfaces. The zero flux condition formally implies the conservation of mass. The constraint $u \geq 0$ and the conditions (1.4) set up an obstacle-type free boundary problem. Summarizing, nonnegative strong solutions of the Cauchy problem are also solutions of the obstacle-type free boundary problem if $0 < n < 3$. A natural idea to investigate this problem for small n is to consider the limit as $n \to 0^+$. However, for $n = 0$, (1.1) becomes the linear equation (1.2), whose solutions are C^∞, have infinite speed of propagation, and do not satisfy $u \geq 0$. Hence, as already noted, the limiting problem as $n \to 0^+$ cannot be the Cauchy problem for the linear equation. We expect that the limiting problem is the obstacle-type free boundary problem and prove it in the self-similar case. Anyway, for $n = 0$ the Cauchy problem is very different from the obstacle-type free boundary problem. This distinction is even sharper for negative values of n: (1.1) has a strong tendency to avoid globally positive solutions; however, for $n < 0$ the equation becomes meaningless if u is zero in an open region. Again we refer to section 2 for more details.

A rigorous investigation of the general (non-self-similar) obstacle-type free boundary problem for $n \leq 0$ puts forward a huge task. In this paper we study the above questions in the self-similar case, introducing three types of self-similar structures: one for $n > -4$, which is the same as for $n > 0$, a second one involving exponential functions for $n = -4$, and "backwards" similarity solutions for $n < -4$. The last two types are discussed in section 5. Most of the paper is devoted to the range of values $-4 < n \leq 0$, in which we consider similarity solutions of the form

$$u(x,t) = t^{-\alpha} f(\eta)\,, \quad \eta = xt^{-\beta}\,. \tag{1.5}$$

The mass conservation property (1.3) implies that $\alpha = \beta$ and (1.5) is in fact a source-type solution. Imposing, in addition, that (1.5) is a solution of (1.1) it follows that the similarity exponents are given by

$$\alpha_n = \beta_n = \tfrac{1}{n+4}\ , \tag{1.6}$$

and the *profile* $f = f(\eta)$ is a solution of the ordinary differential equation

$$(f^n f''')' = c\,(\eta f)' \tag{1.7}$$

with $c = \beta_n$. Formula (1.6) shows the critical role played by the value $n = -4$.

As explained in section 2 (see Theorem 2.1), the obstacle-type free boundary problem for solutions of the form (1.5) with $-4 < n \le 0$ is reduced to the following problem for $f = f(\eta)$:

$$\text{(P)} \qquad \begin{cases} f''' = c\,\eta f^{1-n} & \text{for } -a < \eta < a, \\[2ex] f'(0) = f(a) = f'(a) = 0, & \displaystyle\int_{\mathbb{R}} f(\eta)\,d\eta = M, \\[2ex] f \in C^3(-a,a) \cap C^1(\mathbb{R}), & f(\eta) > 0 \quad \text{for } -a < \eta < a, \\[2ex] f(\eta) \text{ even}, & f(\eta) = 0 \quad \text{if } |\eta| > a, \end{cases}$$

where $c = \beta_n$ and a is a positive number to be determined as a part of the problem. We introduce the constant c in order to unify some statements for the three cases $n > -4$, $n = -4$, and $n < -4$.

In [11] the self-similar source-type solutions for $n > 0$ have also the form (1.5)–(1.6) and are characterized by problem (P), with the exception that for $2 < n < 3$ the possibility of noneven solutions is not ruled out so far. The above-mentioned theorems of [11] state precisely that problem (P) has a unique solution if $0 < n < 3$ and has no solution if $n \ge 3$.

The condition $f \in C^3(-a,a)$ can be replaced by $f \in C^\infty(-a,a)$. Notice also that, in the set $\{f > 0\}$, the differential equation in (P) is equivalent to

$$f^n f''' = c\,\eta f\ . \tag{1.8}$$

This shows that the constant of integration for (1.7) is zero; see (2.2).

A preliminary step for considering the limit as $n \to 0^+$ is to establish an existence and uniqueness result for $n = 0$. Since the proof is identical for $n < 0$, $n \ne -4$, we state our first theorem in the following way.

THEOREM A. *Let $c > 0$, $n \le 0$, and $n \ne -4$. Then problem* (P) *has a unique solution f. Furthermore, f is strictly decreasing and $\sqrt{f}$ strictly concave in* $(0,a)$.

This theorem will be proved in section 3. The concavity property, which holds also for $0 < n < 3$, has not been published before to our knowledge. We will prove it for all $n < 3$. In the special case $n = -4$, problem (P) has a solution for some value of $c > 0$ and the solution is not unique (see section 5).

We proceed to state the theorem describing the limiting process as $n \to 0$, which will be proved in section 4.

THEOREM B. *For each n, $0 \le n < 1$, let (f_n, a_n) be the solution of problem* (P) *with $c = \beta_n$. Then as $n \to 0^+$ we have that $a_n \to a_0$ and $f_n \to f_0$ in $C^1(\mathbb{R})$.*

The study of the limiting process as $n \to 0^+$ is closely related to more general results on continuity with respect to the parameter n, which we work out in the range $-4 < n < 3$ (Theorem 4.1).

Let us point out that, for any $n \le 0$, the behavior of a solution f of problem (P) at $\eta = a$ is

$$f(\eta) \sim \tfrac{1}{2}\gamma\,(a-\eta)^2 \quad \text{as } \eta \to a^- , \quad \text{where } \gamma = f''(a^-) > 0. \tag{1.9}$$

(Notice that $f''(a^-)$ is positive, rather than zero, by the standard uniqueness theorem for solutions of $f''' = c\eta f^{1-n}$ when $n \le 0$.) From [11] we know that (1.9) also holds for $0 < n < 3/2$, while $f''(a^-) = +\infty$ for $3/2 \le n < 3$.

The self-similar solutions $u(x,t)$ of the obstacle-type free boundary problem defined for $-4 < n \le 0$ by (1.5)–(1.6) and Theorem A share many properties with the source solutions for $0 < n < 3$. In fact, since f has bounded support, u has finite speed of propagation. The set $\{u > 0\}$ is bounded by the free boundaries $x = \pm a t^{\beta}$ and, as already noted, $u(x,t) \to M\delta(x)$ as $t \to 0^+$. Furthermore, $f \in C^1(\mathbb{R})$, hence the contact angle u_x at the interface is zero, while (1.9) implies that u *is not* C^2 at the contact line. However, there is an important and subtle difference between the positive and the nonpositive values of n. This difference amounts to the distinction between the Cauchy problem and the obstacle-type free boundary problem when $n \le 0$. For $0 < n < 3$ the source solutions are both solutions of the Cauchy problem and of the obstacle-type free boundary problem. On the contrary, for $n = 0$ the function u is *not* a solution of the linear equation (1.2) across the free boundary (as exhibited by (2.4) below), while for $-4 < n < 0$ the product $u^n u_{xxx}$ is meaningless in an open region where $u = 0$. Of course in all cases u is a classical solution of (1.1) in its positivity set. An analysis of these differences will be performed in section 2.

The situation is rather different from that of the heat equation $u_t = u_{xx}$ and the *porous media equation*

$$u_t = (u^n u_x)_x \tag{1.10}$$

in the *fast diffusion range* $n < 0$. For these second-order equations the current problems have globally positive solutions. Hence, they have infinite speed of propagation and the condition $u \ge 0$ does not generate obstacle problems. However, there is also a critical value of n, namely, $n = -2$, which, from the point of view of dimensional analysis, corresponds to $n = -4$ for (1.1). The porous media equation is usually considered for $0 < n < \infty$ (*slow diffusion*, with finite speed of propagation) and for $-1 < n < 0$ (fast diffusion, with infinite speed of propagation). See the surveys of Aronson [1] and Peletier [45]. The range $n \le -1$ has been less studied. Let us mention that (1.10) has globally positive source-type solutions for $-2 < n < 0$ (Barenblatt solutions [2]). See also [33] for the range $-2 < n \le -1$. If $n \le -2$ the Cauchy problem for (1.10) has positive solutions which decay as $|x| \to \infty$, but no positive solution has finite mass [39], [47]. Equation (1.10) in higher dimensions,

$$u_t = \operatorname{div}(u^n\,\nabla u),$$

has also been studied for $n \le -1$ [47], [29], [30]. We refer to [31] and references therein for the special case $n = -1$, $u_t = \Delta \log u$, which arises in several models of fluid mechanics and in differential geometry.

2. The obstacle-type free boundary problem. We proceed to a precise definition of the obstacle-type free boundary problem for (1.1) sketched in the introduction. Let $n \in \mathbb{R}$,

$$u_0 \in H^1(\mathbb{R}), \quad u_0 \ge 0, \quad \text{support}\, u_0 \text{ compact}, \quad M = \int_{\mathbb{R}} u_0(x)\,dx > 0, \tag{2.1}$$

$$Q = \mathbb{R} \times (0, \infty), \qquad \mathcal{P} = \mathcal{P}(u) = \{(x,t) \in Q \ : \ u(x,t) > 0\},$$

where $H^k(\mathbb{R})$ stands for the Sobolev space

$$H^k(\mathbb{R}) \equiv \{v \in L^2(\mathbb{R}) \ : \ v^{(j)} \in L^2(\mathbb{R})\,,\ j = 1, \dots, k\}.$$

We say that $u = u(x,t)$ is a solution of Problem OFB (obstacle-type free boundary) if u satisfies the following seven conditions.

1. $u \geq 0$ in Q.
2. $u \in C(\overline{Q}) \cap C^\infty(\mathcal{P})$ and u_x is continuous in x for almost every $t > 0$.
3. u is a classical solution of (1.1) in $\mathcal{P}$.
4. (Zero flux condition at the interfaces). For all x_0 and almost every $t_0 > 0$ such that $(x_0, t_0) \in \partial\mathcal{P}$

$$\lim_{x \to x_0,\ (x,t_0) \in \mathcal{P}} u^n(x,t_0)\, u_{xxx}(x,t_0) = 0.$$

5. (Zero flux condition at infinity). For almost every $t_0 > 0$

$$\lim_{|x| \to \infty,\ (x,t_0) \in \mathcal{P}} u^n(x,t_0)\, u_{xxx}(x,t_0) = 0.$$

6. u satisfies the mass conservation property (1.3).
7. $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$.

The set $\mathcal{P}$ is to be determined as a part of the problem. The free boundaries are the connected components of $\partial\mathcal{P} \cap Q$. The condition (6) is stated separately because a rigorous derivation of (6) from (4) and (5) would require some knowledge of the structure and regularity of the set $\partial\mathcal{P}$. Under the hypotheses (2.1) we expect that any solution u of Problem OFB has finite speed of propagation, i.e., that

$$\mathcal{P}_T = \{(x,t) \in \mathcal{P} \ : \ 0 < t < T\} \quad \text{is bounded for all } T > 0.$$

Instead of including such a strong requirement in the definition, we have introduced the zero flux condition at infinity. In fact, self-similar solutions of Problem OFB have finite speed of propagation and are uniquely determined (see below), although these are open questions for the general (non-self-similar) problem.

If $0 < n < 3$ and the initial datum u_0 satisfies (2.1), the Cauchy problem for (1.1) has a strong solution with finite speed of propagation, which is also a solution of Problem OFB; see [7], [40], [27], and references therein. Strong solutions of (1.1) are functions that satisfy the definition of weak solution of [9] and, *in addition*, the entropy estimates of [4] and [14]. Some technical details of the definition of Problem OFB are adapted to the current stage of the theory for $n > 0$. New developments (for $n > 0$ or for $n \leq 0$) may lead to simplifications of this definition.

Weak solutions in the sense of [9] exist for all $n > 0$. Strong solutions are known to exist for $0 < n < 3$ and formal asymptotics indicates that they do not exist for $n \geq 3$. Uniqueness for strong solutions is an open problem. The weak solutions of [9] may have nonzero contact angle and, hence, may not be solutions of Problem OFB.

The source solutions of [11] model very closely the behavior of general strong solutions at the free boundary, as explained in [4]. The source solutions are strong solutions only away from $t = 0$, because the Dirac mass is more singular than the H^1 functions. (A study of (1.1) with measures as initial data was performed in [27].)

The estimates of the free boundary obtained in [7], [40], [27], and references therein involve constants which tend to infinity as $n \to 0^+$, for example, the estimates associated with the differential inequalities (5.14) and (5.19) of [7]. Hence the results of these papers *do not imply* that the general limiting problem as $n \to 0$ has finite speed of propagation. This stresses the interest of proving that at least in the particular case of source-type solutions the limit as $n \to 0^+$ indeed has finite speed of propagation (Theorems A and B).

Let us give a precise formulation of the obstacle-type free boundary problem for self-similar solutions of the form (1.5)–(1.6). As already noted, this form implies that the initial datum is $M\delta(x)$. Problem OFB translates into the following problem for $f = f(\eta)$, which we call problem (Q):

1. $f \geq 0$ in $\mathbb{R}$.
2. $f \in C^1(\mathbb{R}) \cap C^\infty(P)$, where $P = \{\eta \in \mathbb{R} \,:\, f(\eta) > 0\}$.
3. f is a classical solution of (1.7) in P, with $c = \beta_n$.
4. (Zero flux condition at the interfaces). For all $\eta_0 \in \partial P$

$$\lim_{\eta \to \eta_0,\, \eta \in P} f^n(\eta)\, f'''(\eta) = 0.$$

5. (Zero flux condition at infinity).

$$\lim_{|\eta| \to \infty,\, \eta \in P} f^n(\eta)\, f'''(\eta) = 0.$$

6. $\int_{\mathbb{R}} f(\eta)\, d\eta = M$.

THEOREM 2.1. *If* $-4 < n \leq 0$*, problem* (Q) *is equivalent to problem* (P) *with* $c = \beta_n$ *.*

Proof. It is clear that a solution of (P) is a solution of (Q). Let us prove the converse. Let f be a solution of (Q). Assume for contradiction that $f > 0$ near $+\infty$. Then for some constant K,

$$f^n f''' = c\eta f + K. \tag{2.2}$$

By the zero flux condition at infinity, $c\eta f(\eta) \to -K$ as $\eta \to +\infty$. Since f has a finite mass M, we have that $K = 0$, $f''' > 0$ near $+\infty$, $\lim_{\eta\to+\infty} f''(\eta) = 0$, and $\lim_{\eta\to+\infty} f'(\eta) = 0$. This successively implies that, near $+\infty$, $f'' < 0$, $f' > 0$, and $f < 0$. This contradiction proves that f has arbitrarily large zeros. (A similar argument applies at $-\infty$.) Hence, the set P is a union of *bounded* open intervals. The zero flux condition at the interfaces implies that $K = 0$ and

$$f^n f''' = c\eta f$$

at each of these intervals. Let (b, a), $0 \leq b < a$, be one of these intervals; hence $f(b) = f'(b) = f(a) = f'(a) = 0$ and $f''(b^+) \geq 0$. But $f''' > 0$ implies $f'' > 0$ in (b, a), which contradicts $f'(b) = f'(a) = 0$. This proves that $b < 0$ and, by a similar argument, $a > 0$. In conclusion, the set P is formed by a unique interval (b, a) with $b < 0 < a$. Finally, we prove that f is even. To that end it is enough to obtain that $f'(0) = 0$. Assume for contradiction that $f'(0) \neq 0$. Replacing, if necessary, $f(\eta)$ by $f(-\eta)$ we may take $f'(0) > 0$. Then the function $v(\eta) = f(\eta) - f(-\eta)$ satisfies

$$v'''(\eta) = c\,\eta \left(f^{1-n}(\eta) - f^{1-n}(-\eta)\right), \quad v(0) = 0, \quad v'(0) > 0, \quad v''(0) = 0\,.$$

This implies that v, v', and v'' are increasing and positive for $0 < \eta < \min\{a, |b|\}$. Hence $|b| < a$ and

$$0 < v'(|b|) = f'(|b|); \quad 0 < v''(|b|) = f''(|b|) - f''(b^+); \quad \text{hence } f''(|b|) > 0.$$

Since f'' is increasing in $(0, a)$, the inequalities $f'(|b|) > 0$ and $f''(|b|) > 0$ contradict $f'(a) = 0$. This contradiction completes the proof. □

Remark 2.1. The above reduction of the obstacle-type free boundary problem (Q) to problem (P) follows arguments used in [11] for $n > 0$. The main difference is the replacement of the condition $\eta f(\eta) \to 0$ as $|\eta| \to \infty$ by the zero flux conditions. It can be checked that both approaches are equivalent for $n \geq 0$. However, the condition $\eta f(\eta) \to 0$ as $|\eta| \to \infty$ does not seem appropriate for $n < 0$.

As sketched in the introduction, there is a fundamental and subtle difference between the positive and the nonpositive values of n, which we proceed to analyze for self-similar solutions. Let $u = u(x,t)$ be the solution of the obstacle-type free boundary problem defined by (1.5)–(1.6) and Theorem A. We consider the range of values $-4 < n < 3/2$, in which, as already noted, (1.9) holds. We begin with the case $n = 0$. At $\eta = \pm a$ the function f'' has a jump and f''' is not defined in the classical sense. In the sense of distributions f''' has Dirac masses at $\eta = \pm a$. Hence, f satisfies the relation

$$f'''(\eta) = \tfrac{1}{4}\eta\, f(\eta) + \gamma\, \delta(\eta + a) - \gamma\, \delta(\eta - a) \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (n = 0), \tag{2.3}$$

where $\mathcal{D}'(\mathbb{R})$ stands for the space of distributions on $\mathbb{R}$. Taking into account that $\delta(xt^{-1/4} - a) = t^{1/4}\, \delta(x - at^{1/4})$, it follows that u satisfies

$$u_t + u_{xxxx} = \gamma\, t^{-3/4}\delta'(x + at^{1/4}) - \gamma\, t^{-3/4}\delta'(x - at^{1/4}) \quad \text{in } \mathcal{D}'(\mathcal{Q}). \tag{2.4}$$

Hence, u_{xxxx} has *dipole singularities at the contact line* for $n = 0$.

Similarly, for any $n \in (-4, 3/2)$ it follows from (1.9) that

$$f'''(\eta) = \beta\eta\, f^{1-n}(\eta) + \gamma\, \delta(\eta + a) - \gamma\, \delta(\eta - a) \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (-4 < n < 3/2). \tag{2.5}$$

An important difference for $0 < n < 3/2$ is that $f^n f'''$ makes sense as the product of a continuous function and a measure. The zeros of f "kill" the Dirac masses and the relation

$$f^n(\eta) f'''(\eta) = \beta\eta\, f(\eta) \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (0 < n < 3/2) \tag{2.6}$$

holds. For $n = 0$ the Dirac masses are not canceled (see (2.3)), while for $n < 0$ the product $f^n f'''$ makes no sense outside the positivity set of f. In conclusion, f is a solution in the sense of distributions of (2.6) and, hence, of (1.7) *on the whole line* when $0 < n < 3/2$, while this is *not true* for $n \leq 0$.

Remark 2.2. For $3/2 \leq n < 3$ the relation (2.6) still holds, $f'' \in L^1(\mathbb{R})$ and $f^n f'''$ makes sense as the product of a C^1 function and the derivative of an L^1 function.

Let us emphasize the deep differences between the Cauchy problem for the linear equation (1.2) and the obstacle-type free boundary problem for $n = 0$. It is well known that the linear equation (1.2) does not have finite speed of propagation. All solutions of (1.2) are C^∞ and even analytic in x for fixed t. Furthermore, if u is a solution of the Cauchy problem for (1.2) and $u(x, 0)$ satisfies (2.1), then u changes sign instantaneously [5]. The fundamental solution U (source solution with $M = 1$) of the linear equation (1.2) is (see, e.g., [10], [46])

$$U(x,t) = t^{-1/4}\, F(x\, t^{-1/4})\,, \tag{2.7}$$

where

$$F(\eta) = \int_{-\infty}^{\infty} e^{-(2\pi\xi)^4}\, e^{2\pi i\eta\xi}\, d\xi\,. \tag{2.8}$$

We have that $\int_{-\infty}^{\infty} F(\eta)\, d\eta = 1$ and $U(x,t) \to \delta(x)$ as $t \to 0^+$. The function $F(\eta)$ is an even solution of the differential equation

$$f''' = \tfrac{1}{4}\eta f \tag{2.9}$$

and changes sign infinitely many times. Additional properties of this fundamental solution are summarized in [10].

Coming back to the function f_0 defined in Theorem A for $n = 0$, an integral representation of $f_0(\eta)$ for $-a_0 < \eta < a_0$ can be obtained as follows. Another even solution of (2.9) is

$$G(\eta) = \int_{-\infty}^{\infty} e^{-(2\pi\xi)^4}\, e^{2\pi\eta\xi}\, d\xi\,, \tag{2.10}$$

and therefore

$$f_0(\eta) = \lambda_1\, F(\eta) + \lambda_2\, G(\eta) \quad \text{if } -a_0 < \eta < a_0 \tag{2.11}$$

for some constants λ_1 and λ_2. The constants a_0, λ_1, and λ_2 are determined by the conditions $f_0(a_0) = f_0'(a_0) = 0$ and $2\int_0^{a_0} f_0(\eta)\, d\eta = M$. A power series expansion for f_0 can be found in [21].

We asserted in the introduction that (1.1) has a strong tendency to avoid globally positive solutions. In [10] there is a precise result in this direction for self-similar solutions: the differential equation $f''' = c\,\eta f^{1-n}$ with $n < 0$ and $c > 0$ has no positive solution defined on the whole real line. This means that every positive solution $f(\eta)$ blows up for a finite $\eta = \eta_0$. Hence any even positive solution of the equation in problem (P) is defined only in a bounded set. This result helps to understand that finite speed of propagation for $n < 0$ is closely related to the constraint $u \geq 0$, in sharp contrast to the properties of the porous media equation and other second-order parabolic equations.

We devote the last paragraphs of this section to address another question put forward in the introduction: Is the limiting problem as $n \to 0^+$ a standard parabolic variational inequality? (See the books [32], [26], and [37] for background material on this question.) Although there are detailed studies on second-order parabolic variational inequalities [23], [37] and fourth-order elliptic variational inequalities [17], [24], [25], [35], [36], the literature on concrete properties for fourth-order parabolic variational inequalities seems to be very scarce [18], [19], [20]. Hence, we have worked out some results that we present without proof, as conjectures to promote the discussion on variational inequalities in the thin film context.

We formulate a standard H^2 variational inequality in the following way: Given $u_0 \in H^2(\mathbb{R})$, $u_0 \geq 0$, find

$$u \in L^2(0,T;H^2(\mathbb{R})) \text{ for all } T > 0 \quad \text{such that} \quad u_t \in L^2(Q), \tag{2.12}$$

$$\int_0^T\!\!\int_{\mathbb{R}} (u_t(v-u) + u_{xx}(v_{xx} - u_{xx}))\, dx\, dt \geq 0 \tag{2.13}$$

for all $v \in L^2(0,T;H^2(\mathbb{R}))$ with $v \geq 0$ and all $T > 0$,

$$u \geq 0 \quad \text{in } Q, \quad \text{and} \quad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}\,. \tag{2.14}$$

This problem has a unique solution that satisfies

$$u_t + u_{xxxx} = \Gamma \qquad \text{in } \mathcal{D}'(\mathcal{Q}), \tag{2.15}$$

where Γ is a nonnegative measure, for example, a Dirac mass located along the free boundaries.

This shows the difference between the above variational inequality and the obstacle-type free boundary problem: the right-hand side of (2.4) is not a measure but a dipole. Moreover, the solution of (2.12)–(2.14) has increasing mass and, if it has finite speed of propagation, the conditions at the free boundary are $u = u_x = u_{xx} = 0$. Nevertheless, this does not exclude the possibility of formulating Problem OFB as a less standard variational inequality.

3. Existence and uniqueness: Theorem A. We consider the following boundary value problem for $g = g(s)$, which is a scaled version of problem (P):

$$\begin{cases} g''' = c\, s g^{1-n} & \text{for } -b < s < b, \\ g'(0) = g(b) = g'(b) = 0, & g(0) = 1, \\ g \in C^3(-b,b) \cap C^1(\mathbb{R}), & g(s) > 0 \quad \text{for } -b < s < b, \\ g(s) \text{ even}, & g(s) = 0 \quad \text{if } |s| > b, \end{cases} \tag{3.1}$$

where b is a positive number to be determined as a part of the problem. It is convenient to state carefully the relation between problems (P) and (3.1). Setting

$$f(\eta) = h g(s), \quad \eta = h^{n/4} s, \quad a = h^{n/4} b, \tag{3.2}$$

we have that for all $n \in \mathbb{R}$, given (f, a), then (g, b) is defined by (3.2) with $h = f(0)$. Conversely, if $n \neq -4$ and (g, b) is given, then (f, a) is defined by (3.2) with h determined by

$$M = \int_{\mathbb{R}} f(\eta)\, d\eta = h^{(n+4)/4} \int_{\mathbb{R}} g(s)\, ds. \tag{3.3}$$

In particular, the problems are equivalent for all $n \neq -4$. (The case $n = -4$ will be considered in section 5.) Hence, Theorem A is implied by the following theorem.

THEOREM 3.1. *For any $c > 0$ and any $n \leq 0$, problem* (3.1) *has a unique solution g. Furthermore, g is strictly decreasing and $\sqrt{g}$ strictly concave in* $(0, b)$.

Proof of existence. We are going to perform a shooting argument. Consider the initial-value problem

$$g''' = c\, s g^{1-n}, \qquad s > 0, \qquad\qquad g(0) = 1, \qquad g'(0) = 0, \qquad g''(0) = -p,$$

where p is the shooting parameter. We look for a value of p such that g_p has a first zero b_p and $g_p'(b_p) = 0$. Consider the disjoint sets

$$S^+ = \{p \in \mathbb{R} \;:\; g_p(s) > 0 \text{ for all } s > 0 \text{ as long as } g_p \text{ exists}\},$$

$$S^- = \{p \in \mathbb{R} \;:\; g_p \text{ has a first positive zero } b_p \text{ and } g_p'(b_p) < 0\}.$$

By continuity in the parameter p both sets are open. Any $p \le 0$ belongs to S^+. We proceed to prove that S^- is nonempty. Take $p > 0$. Then $g_p \le 1$ in some maximal interval $[0, c_p]$. Hence in this interval $g_p''' \le cs$ and

$$g_p(s) \le 1 - \frac{p}{2}s^2 + \frac{c}{24}s^4 \quad \text{if } s \in [0, c_p]. \tag{3.4}$$

Since $g_p''(s) \le -p + cs^2/2$ it follows that $s_p \equiv \sqrt{2p/c} \in [0, c_p]$, and by (3.4)

$$g_p(s_p) \le 1 - \tfrac{5p^2}{6c}\,,$$

which becomes negative for p large enough. Hence there exists p such that g_p has a first zero b_p. If $g_p'(b_p) = 0$, then g_p is the desired solution. And if not the set S^- is nonempty and there exists q such that $q \notin S^+$ and $q \notin S^-$. Therefore, g_q has a first zero b_q and $g_q'(b_q) = 0$. This completes the existence proof. □

Proof of uniqueness. Let (g_1, b_1) and (g_2, b_2) be two *different* solutions and set $v = g_1 - g_2$. Since $v(0) = v'(0) = 0$, we have that $v''(0) \ne 0$ by ODE theory, and we may assume $v''(0) > 0$. Then v, v', and v'' are positive and increasing, $b_2 < b_1$, and a contradiction is obtained as in the final part of the proof of Theorem 2.1. □

Proof of the decreasing property. We just observe that $g' < 0$ in $(0, b)$ because g' is convex and $g'(0) = g'(b) = 0$. □

Proof of the concavity property. Since this property also holds for $0 < n < 3$ and has not been published before (as far as we know), we perform the proof for all $n < 3$. Clearly, the proof also applies to the function f of Theorem A. Setting $\Phi = gg'' - (g')^2/2$ we have that

$$2(\sqrt{g})'' = \frac{d}{ds}\left(\frac{g'}{\sqrt{g}}\right) = g^{-3/2}\,\Phi \quad \text{and} \quad \Phi' = gg''' > 0.$$

Therefore Φ is strictly increasing in $(0, b)$ and $\lim_{s\to b} \Phi(s)$ exists. (Notice that the existence of this limit is not evident for $n > 1$.) Since this limit is equal to $\lim_{s\to b} g(s)g''(s)$, both limits are zero. (If they are not zero, g' is unbounded.) Thus $\Phi < 0$ in $(0, b)$ and $\sqrt{g}$ is strictly concave. □

Remark 3.1. The above existence and uniqueness proofs follow methods of [11] and are included for the sake of completeness.

4. The limit as $n \to 0$ and continuity with respect to n. The study of the limiting process will be based on some estimates well behaved with respect to n. Although we have stated Theorem B for the limit as $n \to 0^+$, we will obtain estimates for $-4 < n < 3$, which will imply continuity properties with respect to n in all this range.

First we introduce a Green's function $G(\eta, t)$ for the third derivative. For simplicity of notation we denote the partial η-derivatives of G by $G'(\eta, t)$, $G''(\eta, t), \ldots$ and define G by

$$\begin{cases} G''' = \delta(\eta - t), & 0 < \eta < a\,,\ 0 < t < a, \\ G'(0,t) = G(a,t) = G'(a,t) = 0, & 0 < t < a. \end{cases} \tag{4.1}$$

This is reduced to the interval $(0, 1)$ by the formula

$$G_a(\eta, t) = a^2\, G_1\left(\tfrac{\eta}{a}, \tfrac{t}{a}\right). \tag{4.2}$$

The Green's function G_1 was already used in [34] and is given explicitly by

$$G_1(\eta,t) = \begin{cases} \frac{1}{2}(1-t)(t-\eta^2) & \text{if } \eta \le t, \\ \frac{1}{2}t(1-\eta)^2 & \text{if } \eta \ge t. \end{cases} \tag{4.3}$$

Notice that G_1 *is positive.*

LEMMA 4.1. *Let $f(\eta)$ be the solution of problem* (P) *with any $n < 3$. Then*

1. (i) $f(\eta) \le f(0)$ *and* (ii) $f(\eta) \ge f(0)(1-\eta/a)^2$ *for all $\eta \in [0,a]$.*
2. (i) $M \le 2af(0)$ *and* (ii) $3M \ge 2af(0)$.

Proof. Item 1 holds true because f is decreasing and $\sqrt{f}$ is concave. In turn, item 2 is implied by item 1 and the relation $2\int_0^a f(\eta)\,d\eta = M$. ∎

LEMMA 4.2. *Let $f(\eta)$ be the solution of problem* (P) *with $-4 < n \le 1$ and $c = \beta = 1/(n+4)$. Then*

$$f(0) \le \frac{1}{B_1(n)}\,M^{4/(n+4)} \quad \textit{and} \quad a \ge \frac{1}{2}B_1(n)\,M^{n/(n+4)}\,, \tag{4.4}$$

$$B_1(n) = (2/3)^{4/(n+4)}\,(24(n+4))^{1/(n+4)}\,. \tag{4.5}$$

Proof. From the definition (4.1) of the Green's function G_a it follows that

$$f(\eta) = \beta\int_0^a G_a(\eta,t)\,t(f(t))^{1-n}\,dt \quad \text{for all } \eta \in [0,a]. \tag{4.6}$$

We take $\eta = 0$. Since G_a is positive, $f(t) \le f(0)$ and $n \le 1$, we deduce that

$$(f(0))^n \le \beta\int_0^a G_a(0,t)\,t\,dt.$$

Notice that this holds true *even if n is negative.* From (4.2)–(4.3) we obtain

$$\int_0^a G_a(0,t)\,t\,dt = a^4\int_0^1 s\,G_1(0,s)\,ds = \tfrac{1}{2}a^4\int_0^1 s^2(1-s)\,ds = \tfrac{1}{24}a^4\,.$$

Hence

$$(f(0))^n \le \tfrac{1}{24(n+4)}\,a^4 \quad \text{if } -4 < n \le 1\,. \tag{4.7}$$

Taking in (4.7) the power $1/4$ (rather than $1/n$, which may be negative or meaningless for $n=0$) and using $3M \ge 2af(0)$ (see Lemma 4.1), we have that

$$3M \ge 2(f(0))^{(n+4)/4}\,(24(n+4))^{1/4}\,,$$

from which the first inequality of (4.4) follows. (Notice that $n+4>0$.) This and $M \le 2af(0)$ imply the second inequality of (4.4). ∎

LEMMA 4.3. *Under the hypotheses of Lemma* 4.2 *we have that*

$$f(0) \ge \frac{1}{B_2(n)}\,M^{4/(n+4)} \quad \textit{and} \quad a \le \frac{3}{2}B_2(n)\,M^{n/(n+4)}\,, \tag{4.8}$$

$$B_2(n) = (64(3-n)(5-2n)(2-n)(n+4))^{1/(n+4)}\,. \tag{4.9}$$

Proof. Recalling once again that G_a is positive, we take $\eta = 0$ in (4.6) and use the concavity inequality $f(t) \geq f(0)(1-t/a)^2$ (see Lemma 4.1), obtaining

$$(f(0))^n \geq \beta \int_0^a G_a(0,t)\, t(1-t/a)^{2-2n}\, dt.$$

We compute this integral by means of (4.2)–(4.3):

$$\begin{aligned}\int_0^a G_a(0,t)\, t(1-t/a)^{2-2n}\, dt &= a^4 \textstyle\int_0^1 G_1(0,s)\, s(1-s)^{2-2n}\, ds \\ &= \tfrac{1}{2} a^4 \textstyle\int_0^1 s^2(1-s)^{3-2n}\, ds = \frac{1}{4(3-n)(5-2n)(2-n)}\, a^4\,.\end{aligned}$$

Therefore

$$(f(0))^n \geq \frac{1}{4(3-n)(5-2n)(2-n)(n+4)}\, a^4 \quad \text{if } -4 < n \leq 1\,. \tag{4.10}$$

This inequality, $M \leq 2af(0)$, and $3M \geq 2af(0)$ imply (4.8)–(4.9). ∎

Now we can proceed with the proof of Theorem B.

Proof of Theorem B. Let (f_n, a_n) be as described in the theorem. Below, C stands for a positive constant independent of n as $n \to 0$, which may be different in different occurrences. (We perform the proof as $n \to 0$ rather than only as $n \to 0^+$.) Since f_n is decreasing in $(0, a_n)$, we derive from Lemmas 4.2 and 4.3 a uniform bound for the third derivative in $(-a_n, a_n)$,

$$|f_n'''(\eta)| \leq \beta_n a_n (f_n(0))^{1-n} \leq C \qquad \text{for } -a_n < \eta < a_n\,.$$

Observing that f_n'' and f_n' have zeros in $[0, a_n]$ it follows that

$$|f_n''(\eta)| \leq C, \qquad |f_n'(\eta)| \leq C, \qquad f_n(\eta) \leq C \qquad \text{for all } \eta \in \mathbb{R}.$$

Thus, there exist a number $a > 0$, a function $f \in C^1(\mathbb{R})$, and a sequence $\{n_k\}$ such that $n_k \to 0$, $a_{n_k} \to a$ and

$$f_{n_k} \to f, \qquad f_{n_k}' \to f' \qquad \text{uniformly in } \mathbb{R}.$$

(The convergence is uniform in the whole $\mathbb{R}$ due to the uniform bound of the supports.) Clearly, f is even and $f(\eta) = 0$ for $\eta > a$. Lemma 4.3 implies that $f(0) > 0$. We recall that uniform convergence, in general, implies only the semicontinuity of the support. However, the concavity inequality (see Lemma 4.1)

$$f_n(\eta) \geq f_n(0)\,(a_n - \eta)^2/a_n^2$$

ensures that $f > 0$ in $[0, a)$ and, hence, $[-a, a]$ is indeed the support of f. It is also clear that f satisfies the relations

$$f'(0) = f(a) = f'(a) = 0, \qquad \int_{\mathbb{R}} f(\eta)\, d\eta = M.$$

Finally, let us consider the differential equation $f_{n_k}''' = \beta_{n_k} \eta f_{n_k}^{1-n_k}$ in $(-a_{n_k}, a_{n_k})$. Since the right-hand side converges uniformly in $\mathbb{R}$ to $(1/4)\eta f$ and $f_{n_k}''' \to f'''$ in $\mathcal{D}'((-a,a))$, it follows that $f''' = (1/4)\eta f$ in $(-a, a)$. Thus, (f, a) is the unique solution (f_0, a_0) of problem (P) with $n = 0$ and $c = \beta_0 = 1/4$. The uniqueness of the limit implies that convergence holds for any sequence $\{n_k\}$ such that $n_k \to 0$. ∎

Remark 4.1. The relation (4.7) becomes an equality for $n = 1$. In fact, as noted in [46], f_1 is given by the explicit formula

$$f_1(\eta) = \tfrac{1}{120}(a_1^2 - \eta^2)^2 \quad \text{with} \quad a_1^5 = 225M.$$

Remark 4.2. The inequalities (4.7) and (4.10) imply

$$2 \cdot 6^{1/4} \le a_0 \le 2 \cdot 30^{1/4}\,.$$

Remark 4.3. In Lemma 4.3 we observe that $B_2(n) \to 0$ as $n \to -4^+$. Hence, as $n \to -4^+$ we have that $f_n(0) \to +\infty$, $a_n \to 0$, $f_n(\eta) \to M\delta(\eta)$, and $u_n(x,t) \to M\delta(x)$ for all $t > 0$. This phenomenon is related to the critical role played by the value $n = -4$ from the point of view of dimensional analysis. A similar phenomenon occurs as $n \to 3^-$ (see [8]) at the borderline between existence and nonexistence of solution. The value $n = 3$ is not detected as critical by dimensional analysis.

The above proof suggests a path towards general results on continuity with respect to the parameter n. We proceed to obtain such results in the full range $-4 < n < 3$.

THEOREM 4.1. *For each $n \in (-4,3)$, let (f_n, a_n) be the solution of problem* (P) *with $c = \beta_n$. Then*

1. *a_n is a continuous function of n for $-4 < n < 3$.*
2. *For all $n^* \in (-4,3)$ we have that $f_n \to f_{n^*}$ in $C^1(\mathbb{R})$ as $n \to n^*$.*

If $-4 < n^* < 1$ the proof is identical to the above proof of Theorem B. In the case $1 \le n^* < 3$ we will perform the proof after two lemmas extending the estimates of Lemmas 4.2 and 4.3 to the range $1 \le n < 3$.

LEMMA 4.4. *Let $f(\eta)$ be the solution of problem* (P) *with $1 \le n < 3$ and $c = \beta = 1/(n+4)$. Then*

$$f(0) \ge \frac{1}{B_3(n)}\, M^{4/(n+4)} \quad \textit{and} \quad a \le \frac{3}{2} B_3(n)\, M^{n/(n+4)}\,,$$

$$B_3(n) = 2^{4/(n+4)}\,(24(n+4))^{1/(n+4)}\,.$$

Proof. It is very similar to that of Lemma 4.2. Since now $1-n \le 0$, the inequality (4.7) is reversed. Then $M \le 2af(0)$ and $3M \ge 2af(0)$ are used. ◻

LEMMA 4.5. *Let $f(\eta)$ be the solution of problem* (P) *with $1 \le n < 3$ and $c = \beta = 1/(n+4)$. Then*

$$f(0) \le \frac{1}{B_4(n)}\, M^{4/(n+4)} \quad \textit{and} \quad a \ge \frac{1}{2} B_4(n)\, M^{n/(n+4)}\,,$$

$$B_4(n) = \left[(2/3)^4\, 24^{1-n}\,(n+4)\, n^{-3n}\,(3(3+n)(3-n))^n\right]^{1/(n+4)}\,.$$

Proof. The integral arising in the proof of Lemma 4.3 is divergent if $n \ge 2$. Hence, we follow a different method inspired by [34]. By (4.6) and the monotonicity of f,

$$f(\eta) \ge \beta \int_\eta^a G_a(\eta,t)\, t(f(t))^{1-n}\, dt \ge \beta(f(\eta))^{1-n} \int_\eta^a G_a(\eta,t)\, t\, dt\,.$$

An explicit computation with formula (4.3) shows that

$$\int_\eta^1 G_1(\eta,t)\, t\, dt = \tfrac{1}{24}(1-\eta)^3\,(4\eta^2 + 3\eta + 1) \ge \tfrac{1}{24}(1-\eta)^3\,. \tag{4.11}$$

From these relations and (4.2) we obtain the fundamental *lower bound*:

$$f(\eta) \geq (24(n+4))^{-1/n}\, a^{1/n}(a-\eta)^{3/n} \quad \text{if } 0<\eta<a \quad (1 \leq n < 3). \tag{4.12}$$

This gives an upper bound for $(f(t))^{1-n}$ in (4.6). Using this upper bound and setting, in addition, $\eta = 0$ in (4.6), it follows that

$$f(0) \leq \frac{n^3}{3(3+n)(3-n)\, 24^{(1-n)/n}\,(n+4)^{1/n}}\; a^{4/n}\,. \tag{4.13}$$

The proof is completed by using the relations $3M \geq 2af(0)$ and $M \leq 2af(0)$. $\square$

Proof of Theorem 4.1 *for* $1 \leq n^* < 3$. An important difference with respect to the above proof of Theorem B is that now f''' is unbounded and, if $n \geq 3/2$, also f'' is unbounded. We obtain alternative estimates from the lower bound (4.12), which implies

$$f_n'''(\eta) \leq C(a_n - \eta)^{(3/n)-3} \qquad \text{for all } \eta \in [0, a_n).$$

Hence $|f''(\eta)| \leq C$ if $n < 3/2$, $|f''(\eta)| \leq C|\log(a_n - \eta)|$ if $n = 3/2$, and

$$f_n''(\eta) \leq C(a_n - \eta)^{(3/n)-2} \qquad \text{if } 3/2 < n < 3.$$

If $n = 3/2$ the derivative f_n'' is bounded in, say, $L^2(\mathbb{R})$. If $3/2 < n < 3$ we find that f_n'' is bounded in $L^p(\mathbb{R})$ for $p < n/(2n-3)$. (Notice that $n/(2n-3) > 1$ if $3/2 < n < 3$.) Hence, by Sobolev's imbedding there exists a function $f \in C^1(\mathbb{R})$ and a sequence $\{n_k\}$ such that $n_k \to n^*$ and

$$f_{n_k} \to f, \qquad f'_{n_k} \to f' \qquad \text{uniformly in } \mathbb{R}.$$

Now the proof is completed as in the proof of Theorem B. $\square$

5. Source-type solutions for $n \leq -4$. In this section we will use Theorem A to construct self-similar solutions of obstacle-type free boundary problems associated with (1.1) for $n \leq -4$. See also the recent work of [21], where these solutions are constructed numerically.

For $n < -4$ we consider the self-similar function of the *backwards form*

$$u(x,t) = (T-t)^{-\alpha} f(\eta), \quad \eta = \frac{x}{(T-t)^{\beta}}\,, \tag{5.1}$$

where T is any given real number, the similarity exponents α and β are again given by (1.6), and f is the unique solution of problem (P) with $c = -\beta > 0$. This function u is defined for all $x \in \mathbb{R}$ and all $t < T$, is a solution of (1.1) in its positivity set

$$\{(x,t) \in \mathbb{R}^2 :\, -a(T-t)^{\beta} < x < a(T-t)^{\beta},\; t < T\},$$

and satisfies conditions (1.4) at the contact lines $x = \pm a(T-t)^{\beta}$, which are defined for $-\infty < t < T$ and "blow up" (tend to infinity) as $t \to T^-$. Now conservation of mass reads

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} f(\eta)\,d\eta = M \quad \text{for all } t < T. \tag{5.2}$$

The self-similar solutions (5.1) do not approach a Dirac mass as $t \to 0^+$. However,

$$u(\cdot,t) \to M\delta \quad \text{as } t \to -\infty. \tag{5.3}$$

Thus, we may still talk of source-type solutions in some sense.

The backwards self-similar solutions (5.1) vanish everywhere as they approach a finite time $t = T$. We will denote T as the *extinction time*. These solutions are not completely determined by the mass M. Indeed, a translation in time of a solution produces a new solution. Thus, though the profile f is determined by M, in order to specify completely the self-similar solution we also have to prescribe the extinction time T. It is worth observing that, although $u(\cdot, t) \to 0$ uniformly as $t \to T^-$, (5.2) shows that there is no convergence in L^1. This phenomenon also occurs for the forward solutions (1.5)–(1.6) as $t \to +\infty$. However, it is especially noticeable for finite T.

In the critical case $n = -4$ there is no solution in general of problem (P) for a prescribed value $c > 0$. Though there is a solution of problem (3.1), we cannot pass from it to a solution of (P), because the scaling (3.2) fails to produce the required mass (cf. (3.3)). In fact, the mass is invariant under this scaling transformation.

However, given a mass $M > 0$ there exists a value $c > 0$ such that problem (P) has a solution with mass M. To see this we consider

$$f(\eta) = \lambda g(\eta),$$

where g is the solution of problem (3.1) for a certain value of the parameter $c = \hat{c} > 0$, and λ is determined by

$$M = \int_{\mathbb{R}} f(\eta)\, d\eta = \lambda \int_{\mathbb{R}} g(\eta)\, d\eta.$$

The profile f satisfies

$$f'''(\eta) = \hat{c}\lambda^{-4}\eta f^5(\eta).$$

Hence, f is a solution to problem (P) for $c = \hat{c}\lambda^{-4}$. Thus we have proved the following result.

Lemma 5.1. *Let $n = -4$. Given any $M > 0$ there exists a value $c > 0$ such that problem* (P) *has a solution f with mass M.*

Let us observe that such a solution is not unique. Namely, if we consider the scaling transformation (3.2) with arbitrary $h > 0$, we obtain another solution to problem (P) for the same values of c and M.

In this case, $n = -4$, we can construct a self-similar solution of exponential form with a prescribed mass M. Indeed, let f and c be as in Lemma 5.1. Then the function

$$u(x, t) = e^{-ct} f(\eta), \quad \eta = xe^{-ct} \tag{5.4}$$

is a solution of (1.1) in its positivity set

$$\{(x, t) \in \mathbb{R}^2 :\, -ae^{ct} < x < ae^{ct},\ t \in \mathbb{R}\}$$

and satisfies conditions (1.4) at the contact lines $x = \pm ae^{ct}$. In this case (5.4) implies conservation of mass in the form

$$\int_{\mathbb{R}} u(x, t)\, dx = \int_{\mathbb{R}} f(\eta)\, d\eta = M \quad \text{for all } t \in \mathbb{R}\,,$$

and (5.3) holds. Again, we may consider u as a source-type solution in some sense.

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