Stability in a two-dimensional combustion model*

Josephus Hulshof
Faculty of Sciences, Mathematics and Computer Sciences division
Vrije Universiteit, De Boelelaan 1081
1081 HV Amsterdam, The Netherlands
E-mail: jhulshof@cs.vu.nl

Luca Lorenzi
Dipartimento di Matematica, Università di Parma
Via M. D'Azeglio 85/A, 43100 Parma, Italy
E-mail: luca.lorenzi@unipr.it

Abstract

We study the stability of a planar travelling wave in the two-dimensional NEF-combustion model when the reduced Lewis number is equal to zero. The functional analytic setting consists of spaces of suitably exponentially weighted Hölder continuous functions. By exploiting the appearance of the integrated Burgers' equation in the equations for perturbations of the wave we avoid the usual assumption that the perturbation must be localized in the lateral space coordinate.

AMS Subject Classification. 35K05, 35R35, 35K60, 35B65, 35B35, 37C75, 80A25.

Keywords: Free boundary problems, stability of travelling waves, fully nonlinear parabolic problems.

1 Introduction

In the 1980s Sivashinsky introduced and studied a two-dimensional thermo-diffusive model for flames in gaseous mixtures. This model is based on the assumption that the diffusion coefficients of the gas and the temperature are almost equal (Nearly Equidiffusional Flame theory), and on the assumption that the activation energy $\mathcal{Z}$ is very large. With the combustion confined to a thin zone of order $\mathcal{Z}^{-1}$, this leads to the following free boundary problem for the reduced temperature Θ, the reduced enthalpy S, and the front ϕ:

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} = \Delta \Theta, & t>0,\ y\in\mathbb{R},\ \eta<\phi(t,y), \\ \Theta = 1, & t>0,\ y\in\mathbb{R},\ \eta\geq\phi(t,y), \\ \dfrac{\partial S}{\partial t} = \Delta S - \lambda\Delta\Theta, & t>0,\ y\in\mathbb{R},\ \eta\neq\phi(t,y), \\ [\Theta]=[S]=0, & t>0,\ y\in\mathbb{R}, \\ \left[\dfrac{\partial\Theta}{\partial n}\right] = -\exp(S),\ \left[\dfrac{\partial S}{\partial n}\right] = \lambda\left[\dfrac{\partial\Theta}{\partial n}\right], & t>0,\ y\in\mathbb{R}, \end{cases} \tag{1.1}$$

where n denotes the outward normal to the surface $\eta=\phi(t,y)$. The differential equations for Θ and S as well as the boundary conditions are coupled by a real parameter λ which is a dimensionless constant

*Work partially supported by the research project "Analysis and control of deterministic and stochastic evolution equations" of the Ministero dell'Istruzione, dell'Università e della Ricerca (M.I.U.R.) and by the research project: "Research Training Network (RTN): Fronts-Singularities (RTN contract: HPRN-CT-2002-00274)"

proportional to the reduced Lewis number. We refer the reader to the original paper by Sivashinsky [10] for more details on the physical aspects of this model and to [3] for a more mathematical discussion of (1.1).

For any $\lambda \in \mathbb{R}$ problem (1.1) admits a unique (up to translations) planar (i.e. independent of y) travelling wave (TW) solution given by $(t,\eta) \mapsto (\phi(t), \Theta^0_\lambda(\eta+t), S^0_\lambda(\eta+t))$, with

$$\phi(t) = -t, \qquad \begin{cases} \Theta^0_\lambda(x) = \exp(x), & S^0_\lambda(x) = \lambda x \exp(x), \quad \text{if } x \le 0, \\ \Theta^0_\lambda(x) = 1, & S^0_\lambda(x) = 0, \quad \text{if } x \ge 0. \end{cases} \tag{1.2}$$

Stability/instability of this solution under two-dimensional perturbations is of physical relevance. The formal study, made by Sivashinsky in [10], showed that there exists a critical value $\lambda_c < 0$ of λ, such that the planar TW should be orbitally stable for $\lambda \in (\lambda_c, 1)$ and orbitally unstable for $\lambda < \lambda_c$ and for $\lambda > 1$.

The instability of the TW was proved in [3] with respect to small and sufficiently smooth perturbations in the context of Hölder spaces. However, the role of the Kuramoto-Sivashinsky equation, derived in this context by Sivashinsky for the description of cellular instabilities for $\lambda \ge 1$, seems to remain out of the reach of rigorous mathematical methods. We note that whereas in the unstable case the Kuramoto-Sivashinsky equation is expected to play a role, in the stable case this role is played by the integrated Burgers' equation. In the context of bistable reaction-diffusion systems both cases are discussed on a completely formal an heuristic level in [6].

Next, in [2] the first author, in a joint paper, proved stability of the TW in the case where $\lambda = 0$, assuming to perturb *only* the temperature Θ. The quadratic term in the integrated Burgers' equation appearing in the leading order terms in the perturbation analysis is absorbed in the linear terms by means of a Cole-Hopf bifurcation. Remarkably this makes the usual assumption that perturbations are localized in the lateral y-direction redundant, which may be interpreted as an a posteriori validation of the role of the integrated Burgers' equation.

More recently, in [8] the second author, in a joint paper, showed stability of the TW for $\lambda \in (\lambda_c, 1)$ in the context of weighted Hölder spaces in a slightly different model where the nonlinear term $\exp(S)$ is replaced by $f(S) = 1 + S + O(S^k)$ for some $k > 5$. They assume that the weight function depends both on η and y and do not use any explicit form of (part of) the quadratic terms. Here the analogy (of the $\lambda = 0$ case) with [5] in the (diagonal) reaction-diffusion context should be noted, where the localization assumption appears in a different form (in view of the use of Sobolev spaces). We emphasize that such results do not fully generalize the one-dimensional results which allow perturbations converging to a translate of the wave.

Here we generalize the result of [2] to the NEF model with $\lambda = 0$, allowing perturbations on Θ and S as well. We prove that the TW solution is stable with respect to suitable weighted (in x only) smooth perturbations.

To prove our stability results, we transform the problem (by suitable changes of coordinates and unknowns) into an equivalent one of the form

$$\begin{cases} D_t \mathbf{u}(t,x,y) = \mathcal{L}\mathbf{u}(t,x,y) + \mathcal{F}(\mathbf{u}(t,\cdot))(x,y), & t \ge 0, \quad (x,y) \in \mathbb{R}^2_- \\ \mathcal{B}\mathbf{u}(t,\cdot)(y) = \mathcal{G}(\mathbf{u}(t,\cdot))(y), & t \ge 0, \quad y \in \mathbb{R}, \end{cases} \tag{1.3}$$

set in the fixed space domain $\mathbb{R}^2_- := \mathbb{R}_- \times \mathbb{R}$, where $\mathcal{L}$ is a second order operator and $\mathcal{B}$ is a first order boundary operator, both with constant coefficients. The nonlinear and nonlocal operators $\mathcal{F}$ and $\mathcal{G}$ are quadratic near 0. Now, the problem of the stability of the TW for problem (1.1) is transformed into the problem of the stability of the null solution to problem (1.3).

The realization L of the operator $\mathcal{L}$ in the space of all the bounded and continuous functions generates an analytic semigroup and its spectrum consists of all the $\omega \in \mathbb{C}$ such that $\mathrm{Re}\,\omega \le -(\mathrm{Im}\,\omega)^2$. Hence the spectrum of L is contained in the left half-plane and 0 is an accumulation point of eigenvalues. Hence we are in a critical case of stability and we cannot invoke the linearized stability principle to prove our stability results.

Working with a space X of weighted continuous functions (corresponding in our situation to an exponential weight function in the x-variable) allows us to limit the continuous spectrum to the half-line $(-\infty, 0]$, but hereby we remain in the critical case of stability. The key idea to overcome this difficulty

consists in determining a suitable projection, which is not a spectral projection, but enjoys most of the typical properties of a projection, namely we determine a projection P such that

$$\|D_t^{\alpha_1} D_x^{\alpha_2} D_y^{\alpha_3} (I-P) e^{tL}\|_{L(X)} \leq M e^{-\omega t}, \quad \forall t \geq 0 \tag{1.4}$$

for some positive constant ω and for any multindex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where by e^{tL} we denote the (analytic) semigroup generated by the operator L. To construct the projection P we observe that the operator L can be split into the sum of two operators L_1 and L_2 acting, respectively, on the x- and the y-variable, which commute in the resolvent sense. Since 0 is a simple eigenvalue of L_1, we can find out the spectral projection P corresponding to the eigenvalue 0 of L_1. Such a projection is readily seen to satisfy (1.4). To carry out our construction of the projection P, it is essential that L can be split into the sum of two operators commuting in the resolvent sense. As it is pointed out in the appendix, this is the case only if $\lambda = 0$, where the system of differential equations in (1.1) and the operator $\mathcal{L}$ in (1.3) are in diagonal form.

Splitting the solution $\mathbf{u}$ to the initial value problem for (1.3) along $P(X)$ and $(I-P)(X)$ we can write $\mathbf{u}(t,x,y) = r(t,y)(e^x, 0, 0) + \mathbf{w}(t,x,y)$. This position allows us to decouple problem (1.3) into two new problems for the pair $(r, \mathbf{w})$. The differential equation in the first system is a nonlinear Burgers' equation. The second system is set in the stable manifold $(I-P)(X)$ where the semigroup satisfies (1.4). What we expect is that $\mathbf{w}$ and its derivatives decay at least with polynomial rate at infinity, and this is just the case. Hence the asymptotic behaviour of the solution to problem (1.1) is determined by the behaviour near infinity of the solution to the Cauchy problem associated with the nonlinear Burgers' equation. Such an equation contains the term r_y^2 which is critical for the stability. Performing a suitable Cole-Hopf transformation allows us to eliminate this term. We get a new equation for the new unknown q, namely a nonlinear heat equation whose solution exhibits the same decay estimates at infinity as the linear heat equation does. Hence q stays bounded in $\mathbb{R}_+ \times \mathbb{R}$ while its derivatives decay polynomially when t approaches infinity.

Coming back to our original problem (1.1) we can conclude that the solution (Θ, S, ϕ), corresponding to initial data close to the TW, exists globally in time and, in the coordinate system attached to the front, Θ, S and ϕ stay bounded and sufficiently close to the TW, while their space- and time-derivatives decay with polynomial rate at infinity.

2 Reduction to a fixed boundary problem

In this section we transform our problem into an equivalent one of the type (1.3). First of all we fix the boundary by setting $\Theta^1(t,x,y) = \Theta(t, x+\phi(t,y), y)$, $S^1(t,x,y) = S(t, x+\phi(t,y), y)$. Moreover, we set $\varphi(t,y) = \phi(t,y) + t$. Easy computations show that the triplet (Θ^1, S^1, φ) solves the following problem:

$$\Theta_t^1 + \Theta_x^1 = \Delta\Theta^1 + (\varphi_y)^2 \Theta_{xx}^1 - 2\varphi_y \Theta_{xy}^1 + \big(\varphi_t - \varphi_{yy}\big)\Theta_x^1, \quad x < 0, \tag{2.1}$$

$$\Theta^1 = 1, \quad x \geq 0, \tag{2.2}$$

$$S_t^1 + S_x^1 = \Delta S^1 + (\varphi_y)^2 S_{xx}^1 + \big(\varphi_t - \varphi_{yy}\big) S_x^1 - 2\varphi_y S_{xy}^1, \quad x \neq 0, \tag{2.3}$$

$$[\Theta^1] = [S^1] = [S_x^1] = 0, \qquad [\Theta_x^1] = -(1+\varphi_y^2)^{-1/2} e^{S^1}. \tag{2.4}$$

Here $[\,\cdot\,]$ denotes the jump at $x = 0$, and $\Delta = D_x^2 + D_y^2$. To decouple the system we argue as in [3] introducing the new unknowns v and w defined by

$$\begin{aligned} &(i) \quad \Theta^1(t,x,y) = \Theta^0(x) + \Theta_x^0(x)\varphi(t,y) + v(t,x,y); \\ &(ii) \quad S^1(t,x,y) = w(t,x,y), \end{aligned} \tag{2.5}$$

where, here and throughout the paper we write Θ^0 instead of Θ_0^0 (see (1.2)).

Performing the change of unknowns in (2.5), we get an equivalent problem for the triplet (v, w, φ). But taking the jump of both sides of (2.5i) at $x = 0$, and recalling that $[\Theta^1] = [S^1] = 0$, we get

$$\varphi(t, y) = -v(t, 0, y), \quad \forall t > 0, \quad \forall y \in \mathbb{R}. \tag{2.6}$$

Setting $\mathbf{u} = (v, w, h)$, where $h(x, y) = w(-x, y)$ for any $(x, y) \in \overline{\mathbb{R}}^2_- := \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$, and replacing the unknown front by its expression in terms of $\mathbf{u}$ given by (2.6), we get the following problem which is equivalent to (2.1)-(2.4):

$$\begin{cases} \mathbf{u}_t(t, x, y) = \mathcal{L}\mathbf{u}(t, x, y) + \mathcal{F}_0(\mathbf{u}(t, \cdot))(x, y) - v_t(t, 0, y)\Psi(\mathbf{u}(t, \cdot))(x, y), & t \geq 0, \quad (x, y) \in \overline{\mathbb{R}}^2_-, \\ (\mathcal{B}\mathbf{u}(t, \cdot))(y) = \mathcal{G}(\mathbf{u}(t, \cdot))(y), & t \geq 0, \quad y \in \mathbb{R}. \end{cases} \tag{2.7}$$

In (2.7) the second-order differential operator $\mathcal{L}$, the boundary differential operator $\mathcal{B} = (B_0, B_1, B_2)$ and $\mathcal{F}_0(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u}))$ are given, respectively, by

$$\mathcal{L}\mathbf{u} = (\Delta v - v_x, \Delta w - w_x, \Delta h + h_x); \tag{2.8}$$

$$\begin{cases} B_0\mathbf{u} = w(0, \cdot) - h(0, \cdot), \\ B_1\mathbf{u} = w_x(0, \cdot) + h_x(0, \cdot), \\ B_2\mathbf{u} = v(0, \cdot) + h(0, \cdot) - v_x(0, \cdot); \end{cases} \tag{2.9}$$

$$\begin{aligned} f_1(\mathbf{u}) = (v_y(0, \cdot))^2\big(\Theta^0_{xx} - v(0, \cdot)\Theta^0_{xxx} + v_{xx}\big) + 2v_y(0, \cdot)\big(-v_y(0, \cdot)\Theta^0_{xx} + v_{xy}\big) \\ + v_{yy}(0, \cdot)\big(-v(0, \cdot)\Theta^0_{xx} + v_x\big), \end{aligned} \tag{2.10}$$

$$f_2(\mathbf{u}) = (v_y(0, \cdot))^2 w_{xx} + 2v_y(0, \cdot)w_{xy} + v_{yy}(0, \cdot)w_x, \tag{2.11}$$

$$f_3(\mathbf{u}) = (v_y(0, \cdot))^2 h_{xx} - 2v_y(0, \cdot)h_{xy} - v_{yy}(0, \cdot)h_x, \tag{2.12}$$

while $\Psi(\mathbf{u})$ and $\mathcal{G}(\mathbf{u})$ are defined by

$$\Psi(\mathbf{u}) = \big(-v(0, \cdot)\Theta^0_{xx} + v_x, w_x, -h_x\big) \tag{2.13}$$

and

$$\mathcal{G}(\mathbf{u}) = (0, 0, g(\mathbf{u})), \;\; g(\mathbf{u}) = 1 + h(0, \cdot) - (1 + (v_y(0, \cdot))^2)^{-1/2} e^{h(0, \cdot)}. \tag{2.14}$$

Problem (2.7) still contains the unknown $v_t(t, 0, y)$ in its right-hand side. However, evaluating the first component of the differential equation in (2.7) at $x = 0$, we can get $v_t(t, 0, y)$ in terms of the space derivatives of $\mathbf{u}$, provided $v_x(t, 0, y) - v(t, 0, y) \neq -1$. Thus, we finally get the following problem for $\mathbf{u}$:

$$\begin{cases} \mathbf{u}_t(t, x, y) = \mathcal{L}\mathbf{u}(t, x, y) + \mathcal{F}(\mathbf{u}(t, \cdot))(x, y), & t \geq 0, \quad (x, y) \in \overline{\mathbb{R}}^2_-, \\ (\mathcal{B}\mathbf{u}(t, \cdot))(y) = \mathcal{G}(\mathbf{u}(t, \cdot))(y), & t \geq 0, \quad y \in \mathbb{R}, \end{cases} \tag{2.15}$$

where

$$\mathcal{F}(\mathbf{u}(t, \cdot)) = \mathcal{F}_0(\mathbf{u}(t, \cdot)) - \frac{\Delta v(t, 0, \cdot) - v_x(t, 0, \cdot) + f_1(\mathbf{u}(t, \cdot))(0, \cdot)}{1 - v(t, 0, \cdot) + v_x(t, 0, \cdot)}\Psi(\mathbf{u}(t, \cdot)), \quad \forall t > 0. \tag{2.16}$$

Note that the TW solution to the original problem corresponds to the null solution to (2.15) and the solutions close to the TW correspond to small solutions to (2.15).

3 The function spaces

In this section we introduce the Banach spaces we deal throughout this paper. For notation convenience we use bold style to denote vector-valued functions. According to the notations of the previous sections, we set $\mathbb{R}^2_- := \{(x,y) \in \mathbb{R}^2 : x < 0\}$ and by $\overline{\mathbb{R}}^2_-$ we denote its closure in $\mathbb{R}^2$. Moreover, for any $\mathbf{f} : \mathbb{R}^2_- \to \mathbb{R}^3$, $\mathbf{f} = (f_1, g_1, g_2)$ we denote by $\mathbf{f}^\sharp$ the function defined by $\mathbf{f}^\sharp(x,y) = (e^{-x/2}f(x,y), e^{-x/2}g_1(x,y), e^{x/2}g_2(x,y))$.

Definition 3.1. *For any $k \geq 0$ we define the function space X_k by*

$$X_k = \big\{\mathbf{f} : \overline{\mathbb{R}}^2_- \to \mathbb{R}^3 : \mathbf{f}^\sharp \in C^k(\overline{\mathbb{R}}^2_-)\},$$

and we endow it with the norm $\|\mathbf{f}\|_{X_k} = \|\mathbf{f}^\sharp\|_{C^k(\mathbb{R}^2_-)}$, $\mathbf{f} \in X_k$.

Moreover, for any $a, b \in \mathbb{R}_+ \cup \{+\infty\}$, $a < b$, and any $\alpha \in (0,1)$ we denote by $\mathcal{X}_{\alpha/2,\alpha}(a,b)$ and $\mathcal{X}_{1+\alpha/2,2+\alpha}(a,b)$, respectively, the Banach spaces

$$\begin{aligned}\mathcal{X}_{\alpha/2,\alpha}(a,b) \;=\; \big\{&\mathbf{u} : \mathbf{u}(t,\cdot) \in X_\alpha \quad \forall t \in [a,b], \quad \sup_{a<t<b} \|\mathbf{u}(t,\cdot)\|_{X_\alpha} < +\infty,\\ &\mathbf{u}(\cdot,x,y) \in C^{\alpha/2}([a,b]) \quad \forall (x,y) \in \overline{\mathbb{R}}^2_-, \quad \sup_{(x,y)\in\mathbb{R}^2_-} \|\mathbf{u}(\cdot,x,y)\|_{C^{\alpha/2}([a,b])} < +\infty\big\},\end{aligned}$$

$$\mathcal{X}_{1+\alpha/2,2+\alpha}(a,b) = \{\mathbf{u} : D_t^{\alpha_1} D_x^{\alpha_2} D_y^{\alpha_3} \mathbf{u} \in \mathcal{X}_{\alpha/2,\alpha}(a,b) \text{ for } 2\alpha_1 + \alpha_2 + \alpha_3 \leq 2\}.$$

They are normed by

$$\|\mathbf{u}\|_{\mathcal{X}_{\alpha/2,\alpha}(a,b)} = \sup_{a<t<b} \|\mathbf{u}(t,\cdot)\|_{X_\alpha} + \sup_{(x,y)\in\mathbb{R}^2_-} [\mathbf{u}(\cdot,x,y)]_{C^{\alpha/2}([a,b])},$$

$$\|\mathbf{u}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(a,b)} = \sum_{2\alpha_1+\alpha_2+\alpha_3\leq 2} \|D_t^{\alpha_1} D_x^{\alpha_2} D_y^{\alpha_3} \mathbf{u}\|_{\mathcal{X}_{\alpha/2,\alpha}(a,b)}.$$

Definition 3.2. *For any $\alpha \in (0,1)$ and any $a, b \in \mathbb{R}_+ \cup \{+\infty\}$, $a < b$, $C^{(j+\alpha)/2,j+\alpha}([a,b] \times \mathbb{R})$ $(j = 1, 2)$ denotes the usual parabolic Hölder space*

$$\begin{aligned}C^{(j+\alpha)/2,j+\alpha}([a,b]\times\mathbb{R}) \;=\; \big\{&\psi : \psi(t,\cdot) \in C^{j+\alpha}(\mathbb{R}), \quad \sup_{t\in[a,b]} \|\psi(t,\cdot)\|_{C^{j+\alpha}(\mathbb{R})} < +\infty,\\ &\psi(\cdot,y) \in C^{(j+\alpha)/2}([a,b]), \quad \sup_{y\in\mathbb{R}} \|\psi(\cdot,y)\|_{C^{(j+\alpha)/2}([a,b])} < +\infty\big\},\end{aligned}$$

endowed with the norm

$$\|\psi\|_{C^{(j+\alpha)/2,j+\alpha}([a,b]\times\mathbb{R})} = \sup_{t\in[a,b]} \|\psi(t,\cdot)\|_{C^{j+\alpha}(\mathbb{R})} + \sup_{y\in\mathbb{R}} [\psi(\cdot,y)]_{C^{(j+\alpha)/2}([a,b])}, \qquad j = 1, 2.$$

Definition 3.3. *We denote by $\mathcal{X}_q$ the Banach space of all the functions $q : \overline{\mathbb{R}}_+ \times \mathbb{R} \to \mathbb{R}$ such that $t \mapsto q(t,\cdot)$ is continuous in $[0,+\infty)$ with values in $C^2(\mathbb{R})$, is continuously differentiable with values in $C(\mathbb{R})$, bounded with values in $C^{2+\alpha}(\mathbb{R})$ $(\alpha \in (0,1))$. $t \mapsto q_t(t,\cdot)$, and $t \mapsto q_{yy}(t,\cdot)$ are Hölder continuous with exponent $\alpha/2$ and with values in $C(\mathbb{R})$; q_t is bounded with values in $C^\alpha(\mathbb{R})$ and*

$$\begin{aligned}\|q\|_{\mathcal{X}_q} = \;&\sup_{t\geq 0} \|q(t,\cdot)\|_{C(\mathbb{R})} + \sup_{t\geq 0} (1+t)\big(\|q_t(t,\cdot)\|_{C^\alpha(\mathbb{R})} + \|q_{yy}(t,\cdot)\|_{C^\alpha(\mathbb{R})}\big)\\ &+ \sup_{0\leq s<t} (1+s)\left(\frac{\|q_t(t,\cdot) - q_t(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{\alpha/2}} + \frac{\|q_{yy}(t,\cdot) - q_{yy}(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{\alpha/2}}\right) < +\infty.\end{aligned}$$

Moreover, we denote by $\mathcal{X}_{\mathbf{w}}$ the space of all the functions $\mathbf{w} : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^2_- \to \mathbb{R}^3$ such that $t \mapsto \mathbf{w}(t,\cdot)$ is continuous in $[0,+\infty)$ with values in X_2, is continuously differentiable with values in X_0, bounded with

values in $X_{2+\alpha}$; $t \mapsto D_t\mathbf{w}$ and $t \mapsto \mathbf{w}(t,\cdot)$ are Hölder continuous with exponent $\alpha/2$ and with values in X_0 and X_2, respectively, and

$$\begin{aligned}\|\mathbf{w}\|_{\mathcal{X}_{\mathbf{w}}} = & \sup_{t\geq 0}(1+t)\big(\|\mathbf{w}(t,\cdot)\|_{X_{2+\alpha}} + \|\mathbf{w}_t(t,\cdot)\|_{X_\alpha}\big) \\ & + \sup_{0\leq s<t}(1+s)\left(\frac{\|\mathbf{w}(t,\cdot)-\mathbf{w}(s,\cdot)\|_{X_2}}{(t-s)^{\alpha/2}} + \frac{\|D_t\mathbf{w}(t,\cdot)-D_t\mathbf{w}(s,\cdot)\|_{X_0}}{(t-s)^{\alpha/2}}\right) < +\infty.\end{aligned}$$

We conclude this section with the following lemma.

Lemma 3.4. *Let $q : \overline{\mathbb{R}}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$[[q]] := \sup_{t\geq 0}\|q(t,\cdot)\|_{C(\mathbb{R})} + \sup_{t\geq 0}(1+t)\big(\|q_{yy}(t,\cdot)\|_{C^\alpha(\mathbb{R})} + \|q_t(t,\cdot)\|_{C^\alpha(\mathbb{R})}\big) < +\infty.$$

Then,

$$\begin{cases} i) & \sup_{t\geq 0}(1+t)^{1/2}\|q_y(t,\cdot)\|_{C(\mathbb{R})} \leq C[[q]]; \\ ii) & \sup_{0\leq s<t}(1+s)(t-s)^{-(2-j+\alpha)/2}\|D_y^j q(t,\cdot) - D_y^j q(s,\cdot)\|_{C(\mathbb{R})} \leq C[[q]], \qquad j=1,2. \end{cases} \tag{3.1}$$

If $\mathbf{w} : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_- \times \mathbb{R} \to \mathbb{R}$ is such that

$$[[\mathbf{w}]] := \sup_{t\geq 0}(1+t)\big(\|\mathbf{w}(t,\cdot)\|_{X_{2+\alpha}} + \|D_t\mathbf{w}(t,\cdot)\|_{X_\alpha}\big) < +\infty,$$

then

$$\sup_{0\leq s<t}(1+s)(t-s)^{-(2-j+\alpha)/2}\|\mathbf{w}(t,\cdot)-\mathbf{w}(s,\cdot)\|_{X_j} \leq C[[\mathbf{w}]], \qquad j=1,2. \tag{3.2}$$

Here C denotes a positive constant independent of q (resp. of $\mathbf{w}$).

Proof. The proof is based on interpolation inequalities. Estimates (3.1i) and (3.1ii), for $j=1$, have been proved in [2, Lemma 2.5]. To prove (3.1ii), with $j=2$, and (3.2) it suffices to argue as in the proof of the quoted lemma, observing that, for any $\alpha \in (0,1)$, there exists a positive constant C such that

$$\|f''\|_{C(\mathbb{R})} \leq C[f]^{\alpha/2}_{C^\alpha(\mathbb{R})}[f'']^{1-\alpha/2}_{C^\alpha(\mathbb{R})}, \qquad \forall f \in C^{2+\alpha}(\mathbb{R})$$

and

$$\|\mathbf{u}\|_{X_j} \leq C\|\mathbf{u}\|^{(2-j+\alpha)/2}_{X_\alpha}\|\mathbf{u}\|^{(j-\alpha)/2}_{X_{2+\alpha}}, \qquad \forall \mathbf{u} \in X_{2+\alpha}, \quad j=1,2. \qquad \square$$

4 The fully nonlinear problem

This section, the main body of the paper, is devoted to prove that the null solution to problem (2.15) is stable with respect to $X_{2+\alpha}$-perturbation. As we are going to show, we are in a critical case of stability, since the spectrum of the realization L of $\mathcal{L}$ in X_0 is contained in the left half-plane and contains 0 as an accumulation point of eigenvalues. Hence, we cannot apply the linearized stability principle to prove our stability results, since we cannot eliminate the eigenvalue 0 from the spectrum of L by a spectral projection. Nevertheless we can define a suitable projection which is not a spectral projection P, but enjoys most of the typical properties of a projection. In particular, see Theorem 4.3, the restriction of e^{tL} to $(I-P)(X_0)$ gives rise to an (analytic) semigroup of negative type.

To get such a projection we observe that the operator L can be split into the sum of two operators L_1 and L_2 commuting in the resolvent set. 0 is a simple eigenvalue of L_1. Hence, we can define a spectral projection associated with it: this will be our projection P.

4.1 The realization of $\mathcal{L}$ in weighted Hölder spaces

In this subsection we show that the realization L of the differential operator $\mathcal{L}$ in X_0 generates an analytic semigroup, we characterize its domain and spectrum, and the interpolation spaces of order $\alpha/2$, $1/2$ and $1+\alpha/2$ ($\alpha \in (0,1)$). Finally we provide a suitable projection P satisfying (1.4).

To begin with, let us consider the following theorem which has been already proved, in a more general context, in [7].

Theorem 4.1. *The realization L of the operator $\mathcal{L}$ in X_0 generates an analytic semigroup provided we set*

$$D(L) = \left\{ \mathbf{u} \in X_0 \cap \bigcap_{1<q<+\infty} W^{2,q}_{\rm loc}(\overline{\mathbb{R}}^2_-),\ \ \mathcal{L}\mathbf{u} \in X_0,\ \ \mathcal{B}\mathbf{u}(0,y) = \mathbf{0}\ \ \forall y \in \mathbb{R} \right\},$$

where the boundary differential operator $\mathcal{B}$ is given by (2.9). Its spectrum is given by $\sigma(L) = (-\infty, 0]$.

Moreover, for every $\alpha \in (0,1)$ the set equalities

$$D_L(\alpha/2, \infty) = \{\mathbf{u} \in X_\alpha : B_0\mathbf{u} = 0 \ \text{at}\ x = 0\},$$

$$D_L(1+\alpha/2, \infty) = \{\mathbf{u} \in X_{2+\alpha} : \mathcal{B}\mathbf{u} = \mathbf{0},\ B_0 L\mathbf{u} = 0 \ \text{at}\ x = 0\},$$

hold, with equivalence of the respective norms. Finally,

$$\{\mathbf{u} \in X_1 : B_0\mathbf{u} = 0 \ \text{at}\ x = 0\} \subset D_L(1/2, \infty),$$

with continuous embedding.

Proof. See [7, Theorem A.2 & A.3]. □

As Theorem 4.1 shows we are in a critical case of stability, since the spectrum of L is contained in the left half-plane and 0 is an accumulation point of eigenvalues. To construct a suitable projection satisfying (1.4) we begin by splitting operator $\mathcal{L}$ into the sum of the two operators $L_1 : D(L_1) \subset X_0 \to X_0$ and $L_2 : D(L_2) \subset X_0 \to X_0$ formally defined by $L_1\mathbf{u} = (v_{xx} - v_x, w_{xx} - w_x, h_{xx} + h_x)$ and $L_2\mathbf{u} = (v_{yy}, w_{yy}, h_{yy})$, where

$$D(L_1) = \{\mathbf{u} : D_x^j\mathbf{u} \in X_0 \ \text{for}\ j = 0,1,2,\ \mathcal{B}\mathbf{u} = \mathbf{0}\}, \qquad D(L_2) = \{\mathbf{u} : D_y^j\mathbf{u} \in X_0 \ \text{for}\ j = 0,1,2\}. \tag{4.3}$$

Let us consider the following lemma.

Lemma 4.2. *Both the operators L_1 and L_2 are infinitesimal generators of analytic semigroups in X_0 provided their domains are chosen as in (4.3); $\sigma(L_1) = (-\infty, -1/4] \cup \{0\}$ and $\sigma(L_2) = (-\infty, 0]$. Moreover, for any $\mathbf{f} = (f,g,k) \in X_0$ it holds that*

$$\begin{aligned}
[R(\omega, L_1)\mathbf{f}]_1 &= \frac{1}{\sqrt{1+4\omega}} \left(a(\mathbf{f}) e^{\mu_2 x} - \int_0^x e^{\mu_2 t} f(x-t,y)\,dt + \int_{-\infty}^0 e^{-\mu_1 t} f(t+x,y)\,dt \right), \\
[R(\omega, L_1)\mathbf{f}]_2 &= \frac{1}{\sqrt{1+4\omega}} \left(b(\mathbf{f}) e^{\mu_2 x} - \int_0^x e^{\mu_2 t} g(x-t,y)\,dt + \int_{-\infty}^0 e^{-\mu_1 t} g(t+x,y)\,dt \right), \\
[R(\omega, L_1)\mathbf{f}]_3 &= \frac{1}{\sqrt{1+4\omega}} \left(c(\mathbf{f}) e^{-\mu_1 x} - \int_0^x e^{-\mu_1 t} k(x-t,y)\,dt + \int_{-\infty}^0 e^{\mu_2 t} k(t+x,y)\,dt \right),
\end{aligned} \tag{4.4}$$

where, $\mu_j = \frac{1}{2} + (-1)^j \frac{1}{2}\sqrt{1+4\omega}$ for $j = 1,2$ and

$$a(\mathbf{f}) = \frac{1}{\mu_2 - 1} \left(\mu_2 \int_{-\infty}^0 e^{-\mu_1 t} f(t,y)\,dt + \int_{-\infty}^0 e^{-\mu_1 t} g(t,y)\,dt + \int_{-\infty}^0 e^{\mu_2 t} k(t,y)\,dt \right), \tag{4.5}$$

$$b(\mathbf{f}) = \int_{-\infty}^{0} e^{\mu_2 t} k(t,y)dt, \qquad c(\mathbf{f}) = \int_{-\infty}^{0} e^{-\mu_1 t} g(t,y)dt. \tag{4.6}$$

Finally, $\omega = 0$ is a simple eigenvalue of L_1; the corresponding eigenspace is spanned by the function $\mathbf{U}_0 = (\Theta^0, 0, 0)$ and the spectral projection associated with it is the operator $P : X_0 \to X_0$ defined by $P\mathbf{u} = M(\mathbf{u})\mathbf{U}_0$, where

$$M(\mathbf{u})(y) = \int_{-\infty}^{0} v(x,y)dx + \int_{-\infty}^{0} w(x,y)dx + \int_{-\infty}^{0} e^x h(x,y)dx, \qquad \forall y \in \mathbb{R}. \tag{4.7}$$

Proof. The proof being straightforward, is left to the reader. □

We are now going to prove that, even if P is not a spectral projection associated with the operator L, it enjoys most of typical properties of a projection. In particular, P commutes with e^{tL}, and $(I-P)e^{tL}$ decays with exponential rate as t tends to $+\infty$. This property will play a crucial role in proving our stability results.

Theorem 4.3. *The operator P defined in Lemma 4.2 commutes with the semigroup e^{tL}. Moreover, for any $\varepsilon > 0$ there exists a positive constant $M := M(\varepsilon)$ such that*

$$\|(I-P)e^{tL}\|_{L(X_0)} \le Me^{(-1/4+\varepsilon)t}. \tag{4.8}$$

In particular, (1.4) holds true for any space- and time-derivative of e^{tL}.

Proof. We begin the proof by observing that $e^{tL} = e^{tL_1}e^{tL_2}$ for any $t \ge 0$. To check the previous property we can limit ourselves to proving that all the assumptions of Da Prato-Grisvard's Theorem (see [4]) are satisfied. Hence, we need to check that L_1 and L_2 commute in the resolvent sense. Of course, thanks to Lemma A.1, we can limit ourselves to showing that the relationship $L_2R(\omega, L_1)\mathbf{f} = R(\omega, L_1)L_2\mathbf{f}$ holds for any $\omega \in \rho(L_1)$ and any $\mathbf{f} \in D(L_2)$ and this follows immediately if we take (4.4)–(4.6) into account.

Now, recalling that P is the spectral projection associated with the simple eigenvalue $\omega = 0$ of the operator L_1 we can write

$$P\mathbf{u} = \frac{1}{2\pi i}\int_\gamma R(\omega, L_1)\mathbf{u}\, d\omega, \qquad \forall \mathbf{u} \in X_0, \tag{4.9}$$

where γ is a suitable closed and smooth curve around $\omega = 0$ contained in $\rho(L_1)$, oriented counterclockwise.

Fix $\omega_0 \in \rho(L_2)$. Applying $R(\omega_0, L_2)$ to both sides of (4.9) and taking into account the fact that L_1 and L_2 commute in the resolvent sense, we can easily show that P commutes with $R(\omega_0, L_2)$ and, consequently it commutes with e^{tL_2}. This is enough for our aims. Indeed, $e^{tL} = e^{tL_1}e^{tL_2}$ for any $t > 0$. Hence,

$$Pe^{tL} = Pe^{tL_1}e^{tL_2} = e^{tL_1}Pe^{tL_2} = e^{tL_1}e^{tL_2}P = e^{tL}P,$$

so that P commutes with the semigroup e^{tL}. Moreover, since $\omega = 0$ is an isolated simple eigenvalue of L_1 and $\sup\{\mathrm{Re}\,\omega : \omega \in \sigma(L_1),\ \omega \ne 0\} = -1/4$, it follows that for any $\varepsilon > 0$ there exists a positive constant $M := M(\varepsilon)$ such that

$$\|(I-P)e^{tL_1}\|_{L(X_0)} \le Me^{(\varepsilon-1/4)t}, \qquad \forall t > 0.$$

Since e^{tL_2} is the heat semigroup, then $\|e^{tL_2}\|_{L(X_0)} \le 1$ for any $t \ge 0$. Hence,

$$\|(I-P)e^{tL}\|_{L(X_0)} = \|(I-P)e^{tL_1}e^{tL_2}\|_{L(X_0)} \le \|(I-P)e^{tL_1}\|_{L(X_0)}\|e^{tL_2}\|_{L(X_0)} \le Me^{(\varepsilon-1/4)t},$$

for any $t > 0$, and (4.8) follows. Now observing that for any $t > 0$, $Le^{tL}(I-P) = Le^{L}e^{(t-1)L}(I-P)$, we easily deduce that $Le^{tL}(I-P)$ decays at least as $e^{(\varepsilon-1/4)t}$ as t tends to $+\infty$. The same result holds true for the function $L^n e^{tL}(I-P)$ for any $n \in \mathbb{N}$. Taking into account the characterization of the interpolation spaces $D_L(\alpha/2, \infty)$ and $D_L(1+\alpha/2, \infty)$ in Theorem 4.1, estimate (1.4) immediately follows. □

4.2 A suitable splitting of problem (2.15)

In this subsection we deal with the initial value problem $\mathbf{u}(0,\cdot)=\mathbf{u}_0$ for the nonlinear problem (2.15). To begin with, we split $\mathbf{u}$ along $P(X_0)$ and $(I-P)(X_0)$ as $\mathbf{u}=P\mathbf{u}+\mathbf{w}:=r(t,y)\mathbf{U}_0+\mathbf{w}$. Then, we determine the Cauchy problems satisfied by the pair $(r,\mathbf{w})$. For this purpose we begin by observing that $\mathcal{L}\mathbf{u}=\mathcal{L}P\mathbf{u}+\mathcal{L}\mathbf{w}$. Moreover, $\mathcal{L}P\mathbf{u}=r_{yy}\mathbf{U}_0$ and

$$P\mathcal{L}\mathbf{v}=-(B_2\mathbf{v})\mathbf{U}_0,\qquad \forall\mathbf{v}\in(I-P)(X_2)\ \text{s.t.}\ B_0\mathbf{v}=B_1\mathbf{v}=0. \tag{4.10}$$

Hence, r and $\mathbf{w}$ turn out to solve the following coupled Cauchy problems:

$$i)\ \begin{cases} r_t=r_{yy}-B_2\mathbf{w}+M(\mathcal{F}(\mathbf{u})),\\ r(0,\cdot)=M(\mathbf{u}_0)\end{cases},\qquad ii)\ \begin{cases} D_t\mathbf{w}=\mathcal{L}\mathbf{w}+(B_2\mathbf{w})\mathbf{U}_0+(I-P)\mathcal{F}(\mathbf{u}),\\ \mathcal{B}\mathbf{w}=\mathcal{G}(\mathbf{u}),\\ \mathbf{w}(0,\cdot)=(I-P)\mathbf{u}_0,\end{cases} \tag{4.11}$$

where the linear operator M is defined in (4.7). Straightforward computations and the fact that $\mathcal{B}\mathbf{u}=\mathcal{B}\mathbf{w}=(0,0,g(\mathbf{u}))$ show that

$$\begin{aligned} M(\mathcal{F}(\mathbf{u})) \;=\; &\big(r_y+v_y(0)\big)^2\left(\frac{e^{h(0)}}{\sqrt{1+(r_y+v_y(0))^2}}-h(0)+\frac{v_x(0)-v_{xx}(0)}{1-v(0)+v_x(0)}\int_{-\infty}^0 e^x h(x)dx\right)\\ &+2\big(r_y+v_y(0)\big)\left(\int_{-\infty}^0 e^x h_y(x)dx+\frac{v_y(0)-v_{xy}(0)}{1-v(0)+v_x(0)}\int_{-\infty}^0 e^x h(x)dx\right)\\ &-\frac{v_{xx}(0)-v_x(0)}{1-v(0)+v_x(0)}\int_{-\infty}^0 e^x h(x)dx, \end{aligned}$$

where, to shorten the notation, we simply wrote $D^\alpha v(x)$, $D^\alpha w(x)$ and $D^\alpha h(x)$ ($|\alpha|\le 2$, $x\le 0$) instead of $D^\alpha v(\cdot,x,\cdot)$, $D^\alpha w(\cdot,x,\cdot)$ and $D^\alpha h(\cdot,x,\cdot)$.

Let us now consider the differential equation in (4.11i). Since, as it has been already pointed out, $B_2\mathbf{w}=g(\mathbf{u})$, we can write

$$r_t(t,y)=r_{yy}(t,y)+\tfrac{1}{2}r_y^2+\mathcal{H}(r(t,\cdot),\mathbf{w}(t,\cdot))(y),\qquad \forall y\in\mathbb{R}, \tag{4.12}$$

where $\mathcal{H}(r,\mathbf{w})$ is a *nonlinear* operator which is quadratic near $(0,\mathbf{0})$. The second order terms in the expression of $\mathcal{H}$ are both quadratic in the derivatives of $\mathbf{w}$, and are given by the product of $(\partial^j/\partial y^j)r$ ($j=0,1,2$) multiplied either by y-derivatives of v and h or by $\int_{-\infty}^0 e^x h(x)dx$. Equation (4.12) exhibits a critical growth at 0 due to the presence of the term $\frac{1}{2}r_y^2$. To skip the problems given by the nonlinearity, we perform an Cole-Hopf transformation, namely we set $q:=e^{r/2}-1$. Straightforward computations show that the differential equation (4.12) transforms into the differential equation $q_t(t,y)=q_{yy}(t,y)+$

$\widetilde{\mathcal{H}}(q(t,\cdot),\mathbf{w}(t,\cdot))$ for the unknown $(q,\mathbf{w})$, where

$$\begin{aligned}\widetilde{\mathcal{H}}(q,\mathbf{w}) \;=\; & \frac{q+1}{2}\left(\frac{2q_y}{q+1}+v_y(0)\right)^2 \\ & \times\left(\frac{e^{h(0)}}{\sqrt{1+\left(\frac{2q_y}{q+1}+v_y(0)\right)^2}}-h(0)-1+\frac{v_x(0)-v_{xx}(0)}{1-v(0)+v_x(0)}\int_{-\infty}^0 e^x h(x)dx\right)\\ & +2\left(q_y+\frac{q+1}{2}v_y(0)\right)\left(\int_{-\infty}^0 e^x h_y(x)dx+\frac{v_y(0)-v_{xy}(0)}{1-v(0)+v_x(0)}\int_{-\infty}^0 e^x h(x)dx\right)\\ & -\frac{(q+1)\big(v_{xx}(0)-v_x(0)\big)}{2\big(1-v(0)+v_x(0)\big)}\int_{-\infty}^0 e^x h(x)dx+\frac{q+1}{4}(v_y(0))^2+q_y v_y(0)\\ & +\frac{q+1}{2}\big(e^{h(0)}-1\big)\left(\frac{1}{\sqrt{1+\left(\frac{2q_y}{q+1}+v_y(0)\right)^2}}-1\right)+\frac{q+1}{2}\big(e^{h(0)}-1-h(0)\big)\\ & +\frac{q+1}{2}\left(\frac{1}{\sqrt{1+\left(\frac{2q_y}{q+1}+v_y(0)\right)^2}}-1+\frac{1}{2}\left(\frac{2q_y}{q+1}+v_y(0)\right)^2\right).\end{aligned} \tag{4.13}$$

Let us now observe that

$$\mathcal{K}(q,\mathbf{w}) \;:=B_2\mathbf{w}+(I-P)\mathcal{F}(\mathbf{u}):=\widetilde{g}(q,\mathbf{w})\mathbf{U}_0+(I-P)(\mathcal{F}(2\log(q+1)\mathbf{U}_0+\mathbf{w})), \tag{4.14}$$

where

$$\widetilde{g}(q,\mathbf{w})=1+h(0,\cdot)-\frac{e^{h(0)}}{\sqrt{1+\left(\frac{2q_y}{q+1}+v_y(0)\right)^2}}; \tag{4.15}$$

$$\begin{aligned}& (I-P)\mathcal{F}(2\log(q+1)\mathbf{U}_0+\mathbf{w})\\ =\;& \left(\frac{2q_y}{q+1}+v_y(0)\right)^2\left[\mathbf{w}_{xx}-\frac{v_{xx}(0)-v(0)+1}{1-v(0)+v_x(0)}\widetilde{\mathbf{w}}_x+\left(1+\frac{v_{xx}(0)-v_x(0)}{1-v(0)+v_x(0)}v(0)\right)\mathbf{U}_0\right.\\ & \left.-\left(\frac{e^{h(0)}}{\sqrt{1+\left(\frac{2q_y}{q+1}+v_y(0)\right)^2}}-h(0)-\frac{v_{xx}(0)-v_x(0)}{1-v(0)+v_x(0)}\int_{-\infty}^0 e^x h(x)dx\right)\mathbf{U}_0\right]\\ & +2\left(\frac{2q_y}{q+1}+v_y(0)\right)\left[\widetilde{\mathbf{w}}_{xy}-\frac{v_{xy}(0)-v_y(0)}{1-v(0)+v_x(0)}\widetilde{\mathbf{w}}_x-\frac{v_y(0)-v_{xy}(0)}{1-v(0)+v_x(0)}\left(\int_{-\infty}^0 e^x h(x)dx\right)\mathbf{U}_0\right.\\ & \left.-\left(\int_{-\infty}^0 e^x h_y(x)dx\right)\mathbf{U}_0+\frac{v(0)v_{xy}(0)-v_y(0)(1+v_x(0))}{1-v(0)+v_x(0)}\mathbf{U}_0\right]\\ & -\frac{v_{xx}(0)-v_x(0)}{1-v(0)+v_x(0)}\left[\widetilde{\mathbf{w}}_x-\left(v(0)+\int_{-\infty}^0 e^x h(x)dx\right)\mathbf{U}_0\right],\end{aligned} \tag{4.16}$$

and $\widetilde{\mathbf{w}}=(v,w,-h)$.

Hence, the pair $(q, \mathbf{w})$ turns out to solve the Cauchy problems

$$i)\ \begin{cases} q_t = q_{yy} + \widetilde{\mathcal{H}}(q, \mathbf{w}), \\ q(0,\cdot) = \exp(M(\mathbf{u}_0)/2) - 1 \end{cases}, \qquad ii)\ \begin{cases} D_t\mathbf{w} = \mathcal{L}\mathbf{w} + \mathcal{K}(q, \mathbf{w}), \\ \mathcal{B}\mathbf{w} = \mathcal{G}(\mathbf{u}), \\ \mathbf{w}(0,\cdot) = (I-P)\mathbf{u}_0, \end{cases} \tag{4.17}$$

where the nonlinear operators $\widetilde{\mathcal{H}}$ and $\mathcal{K}$ are given, respectively, by (4.13) and (4.14)–(4.16).

4.3 Optimal regularity for the linear problem

We devote this section to proving optimal Schauder estimates for the solution to the linearized problem associated with system (4.17), namely with the problems

$$i)\ \begin{cases} q_t = q_{yy} + \varphi, \\ q(0,\cdot) = q_0 \end{cases}, \qquad ii)\ \begin{cases} D_t\mathbf{w} = \mathcal{L}\mathbf{w} + \mathbf{f}, \\ \mathcal{B}\mathbf{w} = (0,0,\psi), \\ \mathbf{w}(0,\cdot) = \mathbf{w}_0 \end{cases}. \tag{4.18}$$

Problem (4.18i) has been partly already considered in [2, Theorem 2.6]. Note that the asymptotic estimates of the solution to problem (4.17) are crucial to prove our stability results. However, since not all the estimates we need are contained in the quoted theorem, we go into details.

Theorem 4.4. *Let $\varphi : \overline{\mathbb{R}}_+ \times \mathbb{R} \to \mathbb{R}$ satisfy*

$$[[\varphi]]_0 = \sup_{t\geq 0}(1+t^{1+\alpha/2})\|\varphi(t,\cdot)\|_{C^\alpha(\mathbb{R})} + \sup_{0\leq s<t}(1+s)\frac{\|\varphi(t,\cdot)-\varphi(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{\alpha/2}} < +\infty, \tag{4.19}$$

for some $\alpha \in (0,1)$. Further, assume that $q_0 \in C^{2+\alpha}(\mathbb{R})$. Then, problem (4.18i) admits a unique bounded strict solution q, which belongs to $\mathcal{X}_q$ (see Definition 3.3) and satisfies the following estimate:

$$\|q\|_{\mathcal{X}_q} + \sup_{s\geq 0}(1+s)\frac{\|q_y(t,\cdot)-q_y(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{(1+\alpha)/2}} \leq C\big(\|q_0\|_{C^{2+\alpha}(\mathbb{R})} + [[\varphi]]_0\big), \tag{4.20}$$

for some positive constant C, independent of (q_0, φ).

Proof. Throughout the proof, we denote by C_j ($j \in \mathbb{N}$) positive constants *independent* of the data and t.

As is well-known (see e.g. [9, Theorems 4.3.1 & 4.3.8]), our assumptions guarantee that problem (4.18i) admits a unique strict solution $q \in C^{1+\alpha/2}([0,T]; C(\mathbb{R})) \cap B([0,T]; C^{2+\alpha}(\mathbb{R})) \cap C^{\alpha/2}([0,T]; C^2(\mathbb{R}))$, such that $q_t \in B([0,T]; C^\alpha(\mathbb{R}))$ for any $T > 0$. The solution q is given by the variation-of-constants formula

$$q(t,\cdot) = T(t)q_0 + \int_0^t T(t-s)\varphi(s,\cdot)ds, \qquad \forall t \in [0,T],$$

where $T(t)$ is the Gauss-Weierstrass semigroup, i.e.

$$(T(t)g)(y) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-|y-z|^2/4t} g(z)dz, \qquad \forall t > 0,\ y \in \mathbb{R},$$

for any $g \in C(\mathbb{R})$. Moreover, there exists a positive constant C_1 such that

$$\begin{aligned} &\|q\|_{B([0,1];C^{2+\alpha}(\mathbb{R}))} + \|q_t\|_{B([0,1];C^\alpha(\mathbb{R}))} + \|q_t\|_{C^{\alpha/2}([0,1];C(\mathbb{R}))} + \|q_{yy}\|_{C^{\alpha/2}([0,1];C(\mathbb{R}))} \\ &\leq C_1\|\varphi\|_{B([0,1];C^\alpha(\mathbb{R}))}. \end{aligned} \tag{4.21}$$

It is also well known that for any $\beta \in [0,1)$ and any $k \in \mathbb{N}$ there exist positive constants $C_{\beta,k}$ such that

$$\|D_y^k T(t)g\|_{C(\mathbb{R})} \leq C_{\beta,k}t^{-(k-\beta)/2}\|g\|_{C^\beta(\mathbb{R})}, \qquad \forall t > 0, \quad \forall g \in C^\beta(\mathbb{R}). \tag{4.22}$$

It immediately follows that $T(\cdot)q_0$ belongs to $B([0,+\infty);C^{2+\alpha}(\mathbb{R}))\cap C^1([0,+\infty);C(\mathbb{R}))$ and there exists a positive constant C_2 such that

$$\sup_{t\geq 0}\|T(t)q_0\|_{C(\mathbb{R})}+\sup_{t\geq 0}(1+t)\|D_{yy}T(t)q_0\|_{C^\alpha(\mathbb{R})}\leq C_2\|q_0\|_{C^{2+\alpha}(\mathbb{R})}. \tag{4.23}$$

Let us now consider the convolution term $q_1(t,\cdot)=\int_0^t T(t-s)\varphi(s,\cdot)ds$ and let us estimate q_1 for $t>1$. From (4.22) we immediately deduce that

$$\|q_1(t,\cdot)\|_{C(\mathbb{R})}\leq \sup_{r\geq 0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C(\mathbb{R})}\int_0^{+\infty}(1+s^{1+\alpha/2})^{-1}ds,\qquad \forall t>1, \tag{4.24}$$

since $C_{0,0}=1$.

As far as the second order space derivative of q_1 is concerned, we observe that

$$\begin{aligned}\|D_y^2q_1(t,\cdot)\|_{C(\mathbb{R})}\ &\leq \left\|\int_0^{t/2}D_y^2T(t-s)\varphi(s,\cdot)ds\right\|_{C(\mathbb{R})}+\left\|\int_{t/2}^{t}D_y^2T(t-s)\varphi(s,\cdot)ds\right\|_{C(\mathbb{R})}\\ &\leq C_{0,2}\sup_{r\geq 0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C(\mathbb{R})}\int_0^{t/2}\frac{1}{1+s^{1+\alpha/2}}\frac{1}{t-s}ds\\ &\quad+C_{\alpha,2}\sup_{r\geq 0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})}\int_{t/2}^{t}\frac{1}{1+s^{1+\alpha/2}}\frac{1}{(t-s)^{1-\alpha/2}}ds\\ &\leq C_3\sup_{r\geq 0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C(\mathbb{R})}t^{-1},\qquad \forall t>1.\end{aligned} \tag{4.25}$$

Moreover,

$$\begin{aligned}\|\xi^{1-\alpha/2}D_y^2T(\xi)D_y^2q_1(t,\cdot)\|_{C(\mathbb{R})}&=\xi^{1-\alpha/2}\left\|\int_0^t D_y^4T(t+\xi-s)\varphi(s,\cdot)ds\right\|_{C(\mathbb{R})}\\ &\leq C_{\alpha,4}\sup_{r\geq 0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})}\int_0^{t/2}\frac{\xi^{1-\alpha/2}}{(t+\xi-s)^{2-\alpha/2}}\frac{ds}{1+s^{1+\alpha/2}}\\ &\quad+C_{\alpha,4}\sup_{r\geq 0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})}\int_{t/2}^{t}\frac{\xi^{1-\alpha/2}}{(t+\xi-s)^{2-\alpha/2}}\frac{ds}{1+s^{1+\alpha/2}}\end{aligned}$$

$$\le C_{\alpha,4}\sup_{r\ge0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})}(t/2)^{-(2-\alpha/2)}\int_0^{+\infty}\frac{ds}{1+s^{1+\alpha/2}}$$
$$+C_{\alpha,4}\sup_{r\ge0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})}(t/2)^{-(1+\alpha/2)}\int_0^{+\infty}\frac{ds}{(1+s)^{2-\alpha/2}}$$
$$\le C_4t^{-(1+\alpha/2)}\sup_{r\ge0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})},\qquad\forall\xi\in(0,1).$$

Hence,

$$\sup_{\xi\in(0,1)}\|\xi^{1-\alpha/2}D_y^2T(\xi)D_y^2q_1(t,\cdot)\|_{C(\mathbb{R})}\le C_4t^{-(1+\alpha/2)}\sup_{r\ge0}(1+r^{1+\alpha/2})\|\varphi(r,\cdot)\|_{C^\alpha(\mathbb{R})},\qquad\forall t>1.\quad(4.26)$$

Since $f\in C^\alpha(\mathbb{R})$ if and only if $f\in C(\mathbb{R})$ and $[[f]]_{C^\alpha(\mathbb{R})}:=\sup_{\xi\in(0,1)}\|\xi^{1-\alpha/2}D_y^2T(\xi)f\|_{C(\mathbb{R})}<+\infty$, and the usual norm of $C^\alpha(\mathbb{R})$ is equivalent to the norm $\|\cdot\|_{C(\mathbb{R})}+[[\,\cdot\,]]_{C^\alpha(\mathbb{R})}$, from (4.25) and (4.26) we immediately deduce that

$$\sup_{t\ge1}(1+t)\|D_y^2q_1(t,\cdot)\|_{C^\alpha(\mathbb{R})}\le C_5\big(\|q_0\|_{C^{2+\alpha}(\mathbb{R})}+[[\varphi]]_0\big).\quad(4.27)$$

Taking (4.21), (4.23), (4.24), (4.27) and Lemma A.3 into account, we can now easily show that

$$\sup_{t\ge0}\|q(t,\cdot)\|_{C(\mathbb{R})}+\sup_{t\ge0}(1+t)\|q_{yy}(t,\cdot)\|_{C^\alpha(\mathbb{R})}\le C_6\big(\|q_0\|_{C^{2+\alpha}(\mathbb{R})}+[[\varphi]]_0\big).\quad(4.28)$$

Moreover, since $q_t(t,\cdot)=q_{yy}(t,\cdot)+\varphi(t,\cdot)$, from (4.28) and our assumptions on φ, we easily deduce that

$$\sup_{t\ge0}(1+t)\|D_tq(t,\cdot)\|_{C^\alpha(\mathbb{R})}\le C_7\big(\|q_0\|_{C^{2+\alpha}(\mathbb{R})}+[[\varphi]]_0\big).\quad(4.29)$$

From (4.28), (4.29) and Lemma 3.4 we immediately deduce that $D_y^jq\in C^{(2-j+\alpha)/2}([0,+\infty);C(\mathbb{R}))$ for $j=1,2$ and there exists a positive constant C_8 such that

$$\sup_{0\le s<t}(1+s)\left(\frac{\|D_y^jq(t,\cdot)-D_y^jq(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{(2-j+\alpha)/2}}+\frac{\|D_tq(t,\cdot)-D_tq(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{\alpha/2}}\right)\le C_8\big(\|u_0\|_{C^{2+\alpha}(\mathbb{R})}+[[\varphi]]_0\big),\quad(4.30)$$

for $j=1,2$. Now, (4.20) follows from (4.28)–(4.30). □

We now pass to consider problem (4.18ii). For this purpose we prove the following lemma which provides us a suitable lifting operator mapping $C^k(\mathbb{R})$ into X_{k+1} for any $k\ge0$.

Lemma 4.5. *There exists a lifting operator $\mathcal{N}\in L(C^k(\mathbb{R}),X_{k+1})$ for any $k\ge0$, such that*

i) $\mathcal{B}\mathcal{N}\psi=(0,0,\psi)$, *for any* $\psi\in C(\mathbb{R})$;

ii) $P\mathcal{N}\psi=\mathbf{0}$, *for any* $\psi\in C(\mathbb{R})$;

iii) $P\mathcal{L}\mathcal{N}\psi=-\psi\mathbf{U}_0$, *for any* $\psi\in C^1(\mathbb{R})$.

Proof. Let N be the linear operator defined by

$$N\psi=x\eta(x)\int_{\mathbb{R}}\varphi(\xi)\psi(y+\xi x)d\xi,\qquad\forall\psi\in C(\mathbb{R}),$$

where η is any smooth function satisfying $\eta(x)=1$ for any $x\in[-1,0]$ and $\eta(x)=0$ for any $x\le-2$, while φ is any smooth even function compactly supported in $(-1,1)$ such that $0\le\varphi(x)\le1$ for any $x\in\mathbb{R}$ and with $\|\varphi\|_{L^1(\mathbb{R})}=1$. As is immediately seen, $N\in L(C^k(\mathbb{R}),C^{k+1,\sharp}(\overline{\mathbb{R}}^2_-))$ for any $k\ge0$, where $C^{k+1,\sharp}(\overline{\mathbb{R}}^2_-)$

denotes the set of all the continuous functions $f : \overline{\mathbb{R}}^2_- \to \mathbb{R}$ such that $(x,y) \mapsto e^{-x/2} f(x,y)$ is bounded. Moreover, $(N\psi)(0,\cdot) = 0$ and $(D_x N\psi)(0,\cdot) = \psi$.

Define $\mathcal{N}_0$ by setting

$$\mathcal{N}\psi = (I-P)(-N\psi, 0, 0), \qquad \forall \psi \in C(\mathbb{R}).$$

It is immediate to check that $\mathcal{N} \in L(C^k(\mathbb{R}), X_{k+1})$ for any $k \geq 0$. Moreover, by construction, $\mathcal{N}$ satisfies both properties *i* and *ii)*. Property *iii)* easily follows from *i)* and (4.10). □

The solution to problem (4.18ii), under suitable assumptions on $\mathbf{w}_0$, $\mathbf{f}$ and ψ, is given by the following formula, a variant of the Balakrishnan formula:

$$\mathbf{w}(t,\cdot) = e^{tL}\mathbf{w}_0 + \int_0^t e^{(t-s)L}\big(\mathbf{f}(s,\cdot) + \mathcal{L}\mathcal{N}\psi(s,\cdot)\big)ds - L\int_0^t e^{(t-s)L}\mathcal{N}\psi(s,\cdot)ds, \qquad \forall t > 0. \tag{4.31}$$

To be more precise the following theorem holds.

Theorem 4.6. *Let $\mathbf{w}_0 \in (I-P)(X_{2+\alpha})$, $\psi \in C^{(1+\alpha)/2,1+\alpha}([0,T]\times\mathbb{R})$ and $\mathbf{f} \in \mathcal{X}_{\alpha/2,\alpha}(0,T)$ $(T>0)$, with $\mathbf{f}(t,\cdot) \in (I-P)(X_0)$ for any $t \in [0,T]$, satisfy the compatibility conditions*

$$\mathcal{B}\mathbf{w}_0(0,\cdot) = (0,0,\psi(0,\cdot)), \qquad B_0(\mathcal{L}\mathbf{w}_0(0,\cdot) + \mathbf{f}(0,\cdot)) = 0. \tag{4.32}$$

Then, problem (4.18ii) admits a unique solution $\mathbf{w} \in \mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)$ given by (4.31), where $\mathcal{N}$ is the lifting operator in Lemma 4.5. Moreover, $\mathbf{w}(t,\cdot) \in (I-P)(X_0)$ for any $t \in [0,T]$ and there exists a positive constant $C(T) > 0$, independent of the data, such that

$$\|\mathbf{w}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)} \leq C(T)\Big(\|\mathbf{w}_0\|_{X_{2+\alpha}} + \|\mathbf{f}\|_{\mathcal{X}_{\alpha/2,\alpha}(0,T)} + \|\psi\|_{C^{(1+\alpha)/2,1+\alpha}([0,T]\times\mathbb{R})}\Big). \tag{4.33}$$

Proof. The proof can be easily obtained by adapting the techniques of [7, Theorem 4.1]. □

Due to the particular nonlinearity $\widetilde{\mathcal{H}}$ we are considering, we can assume that $\mathbf{f}$ is split as $\mathbf{f} = \mathbf{g} + \psi\mathbf{U}_0$ for suitable functions $\mathbf{g}$ and ψ.

The following theorem deals with the asymptotic behaviour of the function $\mathbf{w}$ in (4.31).

Theorem 4.7. *Suppose that $\mathbf{g} \in \mathcal{X}_{\alpha/2,\alpha}(0,\infty)$ $(\alpha \in (0,1))$ is such that $\mathbf{g}(t,\cdot) \in (I-P)(X_\alpha)$ for any $t \geq 0$ and*

$$[[\mathbf{g}]]_1 := \sup_{t\geq 0}(1+t)\|\mathbf{g}(t,\cdot)\|_{X_\alpha} + \sup_{0\leq s<t}(1+s)\frac{\|\mathbf{g}(t,\cdot)-\mathbf{g}(s,\cdot)\|_{X_0}}{(t-s)^{\alpha/2}} < +\infty. \tag{4.34}$$

Further, assume that $\mathbf{w}_0 \in (I-P)(X_{2+\alpha})$ and $\psi \in C^{(1+\alpha)/2,1+\alpha}([0,+\infty)\times\mathbb{R})$ is such that

$$\begin{aligned}[[\psi]]_2 = &\sup_{t\geq 0}(1+t)\|\psi(t,\cdot)\|_{C^{1+\alpha}(\mathbb{R})} \\ &+ \sup_{0\leq s<t}(1+s)\left(\frac{\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{(1+\alpha)/2}} + \frac{\|\psi_y(t,\cdot)-\psi_y(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{\alpha/2}}\right) < +\infty\end{aligned} \tag{4.35}$$

and

$$\mathcal{B}\mathbf{w}_0 = (0,0,\psi(0,\cdot)), \qquad B_0(\mathcal{L}\mathbf{w}_0 + \mathbf{g}(0,\cdot)) = (-\psi,0,0). \tag{4.36}$$

Then, problem (4.18ii) with $\mathbf{f} = \mathbf{g} + \psi\mathbf{U}_0$ admits a unique solution $\mathbf{w} \in \mathcal{X}_{\mathbf{w}}$ given by (4.31). Moreover, $\mathbf{w}(t,\cdot) \in (I-P)(X_0)$ for any $t \geq 0$ and

$$\|\mathbf{w}\|_{\mathcal{X}_{\mathbf{w}}} + \sup_{0\leq s<t}(1+s)\frac{\|\mathbf{w}(t,\cdot)-\mathbf{w}(s,\cdot)\|_{X_1}}{(t-s)^{(1+\alpha)/2}} \leq C\left(\|\mathbf{w}_0\|_{X_{2+\alpha}} + [[\mathbf{g}]]_1 + [[\psi]]_2\right). \tag{4.37}$$

Before proving the theorem let us consider the following two lemmas.

Lemma 4.8. *Let Z be a Banach space, let $A : D(A) \subset Z \to Z$ be the generator of an analytic semigroup of negative type in Z, and let $u_0 \in Z$, $f : [0, +\infty) \to Z$ be a $\alpha/2$-Hölder continuous function such that*

$$[[f]]_3 := \sup_{t\geq 0}(1+t)\|f(t)\|_Z + \sup_{0\leq s<t}(1+s)\frac{\|f(t,\cdot)-f(s,\cdot)\|_Z}{(t-s)^{\alpha/2}} < +\infty, \tag{4.38}$$

with $0 < \alpha < 1$. Then, the function

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds \tag{4.39}$$

is the unique classical solution of $u'(t) = Au(t) + f(t)$ such that $u(0) = u_0$; $u'(t,\cdot)$ has values in $D_A(\alpha/2,\infty)$ for any $t > 0$ and

$$\sup_{t\geq 1}(1+t)\big(\|u(t)\|_{D(A)} + \|u'(t)\|_{D_A(\alpha/2,\infty)}\big) + \sup_{1\leq s<t}(1+s)\frac{\|u'(t)-u'(s)\|_Z}{(t-s)^{\alpha/2}} \leq C\left(\|u_0\|_Z + [[f]]_3\right). \tag{4.40}$$

Finally, if $u_0 \in D(A)$ and $Au_0 + f(0) \in D_A(\alpha/2,\infty)$, then $u \in C^{1+\alpha/2}([0,1];Z) \cap C^{\alpha/2}([0,1];D(A))$ and $u' \in B([0,1];D_A(\alpha/2,\infty))$. Moreover, there exists a positive constant C, independent of the data, such that

$$\begin{aligned}&\|u\|_{C^{1+\alpha/2}([0,1];Z)} + \|u\|_{C^{\alpha/2}([0,1];D(A))} + \|u'\|_{B([0,1];D_A(\alpha/2,\infty))}\\ &\leq C\left(\|u_0\|_{D(A)} + \|Au_0+f(0)\|_{D_A(\alpha/2,\infty)} + \|f\|_{C^{\alpha/2}([0,1];Z)}\right).\end{aligned} \tag{4.41}$$

Proof. It is well known (see e.g. [9, Theorem 4.3.8]) that, under our assumptions, formula (4.39) defines the unique classical solution of the equation $u' = Au + f$. Moreover, if $u_0 \in D(A)$ and $Au_0 + f(0) \in D_A(\alpha/2,\infty)$, then $u \in C^{\alpha/2}([0,T];D(A)) \cap C^{1+\alpha/2}([0,T];Z)$ and $u' \in B([0,T];D_A(\alpha/2,\infty))$ for any $T > 0$, and (4.41) is satisfied. Hence, we can limit ourselves to dealing with the asymptotic behaviour of u, simply by checking (4.40). For this purpose we observe that, e^{tA} being a semigroup of negative type, there exist positive constants ω and M_k ($k \in \mathbb{N}$) such that

$$t^k\|A^k e^{tA}\|_{L(Z)} \leq M_k e^{-\omega t}, \qquad \forall t > 0,\ k \in \mathbb{N}. \tag{4.42}$$

Taking (4.42) into account, it is now easy to show that the functions $u_1(t) = e^{tA}u_0$ and Au_1 decay exponentially at infinity. In particular, there exists a positive constant C such that

$$\sup_{t\geq 1}(1+t)(\|u_1(t)\|_{D(A)} + \|u_1'(t)\|_{D_A(\alpha/2,\infty)}) + \sup_{1\leq s<t}(1+s)\frac{\|u_1'(t)-u_1'(s)\|_Z}{(t-s)^{\alpha/2}} \leq C\|u_0\|_Z. \tag{4.43}$$

Let us now consider the integral term $u_2(t) = \int_0^t e^{(t-s)A}f(s)ds$. As is easily seen

$$\begin{aligned}\|u_2(t)\|_Z &\leq \sup_{r\geq 0}(1+r)\|f(r)\|_Z\left(\int_0^{t/2}(1+s)^{-1}e^{-\omega(t-s)}ds + \int_{t/2}^t(1+s)^{-1}e^{-\omega(t-s)}ds\right)\\ &\leq \sup_{r\geq 0}(1+r)\|f(r)\|_Z\left(e^{-\omega t/2}\int_0^{t/2}(1+s)^{-1}ds + (1+t/2)^{-1}\omega^{-1}\right)\\ &\leq C(1+t)^{-1}\sup_{r\geq 0}(1+r)\|f(r)\|_Z, \qquad \forall t > 0.\end{aligned} \tag{4.44}$$

Similarly, since

$$u_2'(t) = \int_0^t Ae^{(t-s)A}\big(f(s)-f(t)\big)ds + e^{tA}f(t), \qquad \forall t > 0,$$

we easily deduce that

$$\begin{aligned}\|u_2'\|_Z &\leq M_1\sup_{0\leq\sigma<\tau}(1+\sigma)\frac{\|f(\tau)-f(\sigma)\|_Z}{(\tau-\sigma)^{\alpha/2}}\int_0^t(1+s)^{-1}(t-s)^{\alpha/2-1}e^{-\omega(t-s)}ds + M_0e^{-\omega t}\sup_{t\geq 0}\|f(t)\|_Z\\ &\leq M_1\left((t/2)^{\alpha/2}e^{-\omega t/2} + (1+t/2)^{-1}\int_0^{+\infty}s^{\alpha/2-1}e^{-\omega s}ds\right)[[f]]_3 + M_0e^{-\omega t}\sup_{t\geq 0}\|f(t)\|_Z\\ &\leq C(1+t)^{-1}[[f]]_3.\end{aligned} \tag{4.45}$$

Moreover, for any $0 \le s < t$, it holds that

$$\begin{aligned} u_2'(t) - u_2'(s) \;&= \int_0^s A\big(e^{(t-\sigma)A} - e^{(s-\sigma)A}\big)\big(f(\sigma) - f(s)\big)d\sigma + \big(e^{tA} - e^{sA}\big)f(s) \\ &\quad + e^{(t-s)A}\big(f(t) - f(s)\big) + \int_s^t Ae^{(t-\sigma)A}\big(f(\sigma) - f(t)\big)d\sigma. \end{aligned} \tag{4.46}$$

Let us now consider the first term in (4.46) and let us observe that

$$\begin{aligned} &\left\| \int_0^s A\big(e^{(t-\sigma)A} - e^{(s-\sigma)A}\big)\big(f(\sigma) - f(s)\big)d\sigma \right\|_Z \\ &= \left\| \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} A^2 e^{\tau A}\big(f(\sigma) - f(s)\big)d\tau \right\|_Z \\ &\le M_2 \sup_{0\le r_1<r_2} (1+r_1)\frac{\|f(r_2) - f(r_1)\|_Z}{(r_2-r_1)^{\alpha/2}} \int_0^s (1+\sigma)^{-1}d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha/2-2}e^{-\omega\tau}d\tau \\ &\le M_2[[f]]_3 \left(e^{-\omega s/2}\int_0^{s/2} d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha/2-2}d\tau + (1+s/2)^{-1}\int_{s/2}^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha/2-2}e^{-\omega\tau}d\tau \right) \\ &\le C[[f]]_3. \end{aligned} \tag{4.47}$$

All the other terms are easily estimated and give estimates similar to (4.47). Hence, from (4.43) and (4.47) we deduce that there exists a positive constant C such that

$$\sup_{1\le s<t} (1+s)\frac{\|u'(t) - u'(s)\|_Z}{(t-s)^{\alpha/2}} \le C\big(\|u_0\|_Z + [[f]]_3\big). \tag{4.48}$$

Finally, let us estimate the asymptotic behaviour of $\|u'(t)\|_{D_A(\alpha/2,\infty)}$. For this purpose, fix $\xi \in (0,1)$, $t \ge 1$, and observe that

$$\begin{aligned} \xi^{1-\alpha/2}\|Ae^{\xi A}u_2'(t)\|_Z \;&\le \xi^{1-\alpha/2}\left\| \int_0^t A^2 e^{(t+\xi-s)A}\big(f(s) - f(t)\big)ds \right\|_Z + \xi^{1-\alpha/2}\|Ae^{(t+\xi)A}f(t)\|_Z \\ &\le M_2\xi^{1-\alpha/2} \sup_{0\le\sigma<\tau} (1+\sigma)\frac{\|f(\tau) - f(\sigma)\|_Z}{(\tau-\sigma)^{\alpha/2}} \int_0^t \frac{e^{-\omega(t+\xi-s)}(t-s)^{\alpha/2}}{(t+\xi-s)^2(1+s)}ds \\ &\quad + M_1\xi^{1-\alpha/2}\frac{e^{-\omega(\xi+t)}}{t+\xi}\|f(t)\|_Z \\ &\le M_2\xi^{1-\alpha/2}[[f]]_3 \left(\frac{t^{\alpha/2}e^{-\omega(\xi+t/2)}}{(\xi+t/2)^2}\int_0^{t/2} \frac{ds}{1+s} + \frac{1}{1+t/2}\int_{t/2}^t \frac{ds}{(t+\xi-s)^{2-\alpha/2}} \right) \\ &\quad + M_1\xi^{1-\alpha/2}\frac{e^{-\omega(\xi+t)}}{t+\xi}\|f(t)\|_Z. \end{aligned} \tag{4.49}$$

Taking the supremum over all $\xi \in (0,1)$ of the first and the last side of (4.49) gives

$$\sup_{\xi\in(0,1)} \xi^{1-\alpha/2}\|Ae^{\xi A}u_2'(t)\|_Z \le C(1+t)^{-1}[[f]]_3.$$

Since $\|u_2'(t)\|_{D_A(\alpha/2,\infty)} = \|u_2(t)\|_Z + \sup_{\xi\in(0,1)} \xi^{1-\alpha/2}\|Ae^{\xi A}u_2'(t)\|_Z$, we get

$$\|u_2'(t)\|_{D_A(\alpha/2,\infty)} \le C(1+t)^{-1}[[f]]_3. \tag{4.50}$$

The assertion now follows from (4.43)–(4.45), (4.48) and (4.50) and our assumptions on f, since $u' = Au + f$. □

Next we deal with the function

$$\mathbf{u}(t,\cdot)=L\int_0^t e^{(t-s)L}\mathcal{N}\psi(s,\cdot)ds, \tag{4.51}$$

where $\mathcal{N}$ is the lifting operator defined in Lemma 4.5.

Lemma 4.9. *Suppose that $\psi\in C^{(1+\alpha)/2,1+\alpha}([0,+\infty)\times\mathbb{R})$ $(\alpha\in(0,1))$ is such that*

$$[[\psi]]_4=\sup_{t\ge0}(1+t)\|\psi(t,\cdot)\|_{C^{1+\alpha}(\mathbb{R})}+\sup_{0\le s<t}(1+s)\frac{\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{(1+\alpha)/2}}.$$

Then, the function $\mathbf{u}$ given by (4.51) belongs to $\mathcal{X}_{1+\alpha/2,2+\alpha}(1,\infty)$. Moreover, there exists a positive constant C, independent of ψ, such that

$$\begin{aligned}&\sup_{t\ge1}(1+t)\big(\|\mathbf{u}(t,\cdot)\|_{X_{2+\alpha}}+\|D_t\mathbf{u}(t,\cdot)\|_{X_\alpha}\big)+\sup_{1\le s<t}(1+s)\frac{\|\mathbf{u}(t,\cdot)-\mathbf{u}(s,\cdot)\|_{X_2}}{(t-s)^{\alpha/2}}\\&+\sup_{1\le s<t}(1+s)\left(\frac{\|\mathbf{u}(t,\cdot)-\mathbf{u}(s,\cdot)\|_{X_1}}{(t-s)^{(1+\alpha)/2}}+\frac{\|D_t\mathbf{u}(t,\cdot)-D_t\mathbf{u}(s,\cdot)\|_{X_0}}{(t-s)^{\alpha/2}}\right)\\&\le C[[\psi]]_4.\end{aligned} \tag{4.52}$$

Proof. Throughout the proof, we denote by C_j $(j\in\mathbb{N})$ positive constants, *independent* of the data and t.

By assumptions $\psi\in C^{(1+\alpha)/2,1+\alpha}([0,+\infty)\times\mathbb{R})$. Hence, $\mathcal{N}\psi\in B([0,+\infty),X_{2+\alpha})$ (cf. Lemma 4.5). Moreover, there exists a positive constant C_1 such that

$$\sup_{t\ge0}(1+t)\|\mathcal{N}\psi(t,\cdot)\|_{X_{2+\alpha}}\le C_1\sup_{t\ge0}(1+t)\|\psi(t,\cdot)\|_{C^{1+\alpha}(\mathbb{R})}. \tag{4.53}$$

From Lemma 4.5 we deduce that

$$(1+s)\|\mathcal{N}\psi(t,\cdot)-\mathcal{N}\psi(s,\cdot)\|_{X_1}\le C_2(1+s)\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C(\mathbb{R})}\le C_2[[\psi]]_4(t-s)^{(1+\alpha)/2}, \tag{4.54}$$

for any $0\le s<t$ and some positive constant C_2. Hence, $\mathcal{N}\psi$ belongs to $C^{(1+\alpha)/2}([0,+\infty),X_1)$. Since $\big\{\mathbf{f}\in(I-P)(X_1):B_0\mathbf{f}=0 \text{ at } x=0\big\}$ is continuously embedded in $Z:=(I-P)(D_L(1/2,\infty))$ endowed with the norm of $D_L(1/2,\infty)$, (cf. Theorem 4.1) and $\mathcal{N}\psi(t,\cdot)\in(I-P)(X_0)$ for any $t\ge0$, it follows that $\mathcal{N}\psi\in C^{(1+\alpha)/2}([0,+\infty),Z)$. Moreover, from (4.53) and (4.54) we deduce that there exists a positive constant C_3 such that

$$\sup_{t\ge0}\|\mathcal{N}\psi(t,\cdot)\|_{D_L(1/2,\infty)}+\sup_{0\le s<t}(1+s)\frac{\|\mathcal{N}\psi(t,\cdot)-\mathcal{N}\psi(s,\cdot)\|_{D_L(1/2,\infty)}}{(t-s)^{(1+\alpha)/2}}\le C_3[[\psi]]_4. \tag{4.55}$$

Let us observe that the function $\mathcal{N}\psi$ satisfies all the assumptions of Lemma 4.8 (see Lemma 4.5), and the restriction of e^{tL} to Z defines an analytic semigroup of negative type (see Theorem 4.3) whose generator is the part of L in Z. Hence, Lemma 4.8 and (4.55) imply that the function $\mathbf{w}(t,\cdot)=\int_0^t e^{(t-s)L}\mathcal{N}\psi(s,\cdot)ds$ belongs to Z for any $t\ge0$, and satisfies the following estimate:

$$\sup_{t\ge1}(1+t)\big(\|L\mathbf{w}(t,\cdot)\|_{D_L(1/2,\infty)}+\|D_t\mathbf{w}(t,\cdot)\|_{D_L(1+\alpha/2,\infty)}\big)\le C_4[[\psi]]_4. \tag{4.56}$$

Let us prove that $L\mathbf{w}\in C^{1+\alpha/2}([1,+\infty);X_0)$. For this purpose, we begin by observing that $L\mathbf{w}$ is differentiable with respect to time and $D_t(L\mathbf{w})(t,\cdot)=LD_t\mathbf{w}(t,\cdot)$ for any $t\ge0$. Indeed, since $B([0,+\infty);D_L(1+\alpha/2,\infty))\subset C([0,+\infty);D(L))$, then

$$L\mathbf{w}(t,\cdot)-L\mathbf{w}(s,\cdot)=\int_s^t LD_t\mathbf{w}(\sigma,\cdot)d\sigma,\qquad\forall t,s\ge0.$$

Consequently, $L\mathbf{w}$ is differentiable with respect to t in $[0,+\infty)$ and $D_tL\mathbf{w}=LD_t\mathbf{w}$. Hence, we can write

$$LD_t\mathbf{w}(t,\cdot)=\int_0^t L^2e^{(t-s)L}\big(\mathcal{N}\psi(s,\cdot)-\mathcal{N}\psi(t,\cdot)\big)ds+Le^{tL}\mathcal{N}\psi(t,\cdot). \tag{4.57}$$

We fix now, and for the rest of the proof, a positive constant $\omega \in (0, 1/4)$. Then, by (4.3) we know that there exist positive constants M_k ($k \in \mathbb{N}$) such that

$$\sup_{t>0} t^{k-1/2} \|L^k e^{tL}(I-P)\|_{L(D_L(1/2,\infty),X_0)} \le M_k e^{-\omega t}.$$

From (4.57) it immediately follows

$$\begin{aligned}
\|D_t L\mathbf{w}(t,\cdot)\|_{X_0} &\le \int_0^t \|L^2 e^{(t-s)L}\big(\mathcal{N}\psi(s,\cdot)-\mathcal{N}\psi(t,\cdot)\big)\|_{X_0} ds + \|Le^{tL}\mathcal{N}\psi(t,\cdot)\|_{X_0} \\
&\le M_2 \sup_{0\le\sigma<\tau}(1+\sigma)\frac{\|\psi(\tau,\cdot)-\psi(\sigma,\cdot)\|_{C(\mathbb{R})}}{(\tau-\sigma)^{(1+\alpha)/2}}\int_0^t (1+s)^{-1}(t-s)^{-1+\alpha/2}e^{-\omega(t-s)}ds \\
&\quad + 2M_1 t^{-3/2}e^{-\omega t}\|\psi(t,\cdot)\|_{C(\mathbb{R})} \\
&\le M_2[[\psi]]_4\left(e^{-\omega t/2}\int_0^t s^{\alpha/2-1}ds + (t/2)^{-1}\int_0^t s^{\alpha/2-1}e^{-\omega s}ds\right) + 2M_1 t^{-1}\|\psi(t,\cdot)\|_{C(\mathbb{R})} \\
&\le C_5 t^{-1}[[\psi]]_4,
\end{aligned} \tag{4.58}$$

for any $t \ge 1$. Moreover,

$$\begin{aligned}
&\|D_t L\mathbf{w}(t,\cdot) - D_t L\mathbf{w}(s,\cdot)\|_{X_0} \\
&\le \left\|\int_0^s L^2\big(e^{(t-\sigma)L}-e^{(s-\sigma)L}\big)\big(\mathcal{N}\psi(\sigma,\cdot)-\mathcal{N}\psi(s,\cdot)\big)d\sigma\right\|_{X_0} + \big\|L\big(e^{tL}-e^{sL}\big)\mathcal{N}\psi(s,\cdot)\big\|_{X_0} \\
&\quad + \left\|\int_s^t L^2 e^{(t-\sigma)L}\big(\mathcal{N}\psi(\sigma,\cdot)-\mathcal{N}\psi(t,\cdot)\big)d\sigma\right\|_{X_0} + \|Le^{(t-s)L}\big(\mathcal{N}\psi(t,\cdot)-\mathcal{N}\psi(s,\cdot)\big)\|_{X_0} \\
&\le C_6[[\psi]]_4\left(\int_0^s \frac{(s-\sigma)^{(1+\alpha)/2}}{1+\sigma}d\sigma\int_{s-\sigma}^{t-\sigma}\frac{e^{-\omega\tau}}{\tau^{5/2}}d\tau + \int_s^t \sigma^{-3/2}e^{-\omega\sigma}d\sigma\right. \\
&\qquad \left. + \int_s^t \frac{e^{-\omega(t-\sigma)}}{(t-\sigma)^{1-\alpha/2}}\frac{1}{1+\sigma}d\sigma + \frac{1}{1+s}\|Le^{(t-s)L}\|_{L(D_L(1/2,\infty),X_0)}(t-s)^{(1+\alpha)/2}\right) \\
&\le C_7[[\psi]]_4\left(\int_0^s \frac{e^{-\omega(s-\sigma)}}{1+\sigma}d\sigma\int_{s-\sigma}^{t-\sigma}\frac{1}{\tau^{2-\alpha/2}}d\tau + \frac{1}{1+s}\int_s^t \frac{e^{-\omega(t-\sigma)}}{(t-\sigma)^{1-\alpha/2}}d\sigma\right. \\
&\qquad \left. + \left(\frac{e^{-\omega s}}{s^{(1+\alpha)/2}}+\frac{1}{s+1}\right)(t-s)^{\alpha/2}\right) \\
&\le C_8[[\psi]]_4\left(e^{-\omega s/2}\int_0^{s/2}d\sigma\int_{s-\sigma}^{t-\sigma}\frac{1}{\tau^{2-\alpha/2}}d\tau + \frac{1}{1+s}\int_{s/2}^s d\sigma\int_{s-\sigma}^{t-\sigma}\frac{1}{\tau^{2-\alpha/2}}d\tau\right. \\
&\qquad \left. + e^{-\omega s}(t-s)^{\alpha/2} + \frac{1}{1+s}(t-s)^{\alpha/2}\right) \\
&\le C_9[[\psi]]_4(1+s)^{-1}(t-s)^{\alpha/2}.
\end{aligned} \tag{4.59}$$

for any $0 \le s < t$. Observing that $\mathbf{u} = L\mathbf{w}$, (4.56), (4.58) and (4.59) imply that $D_t\mathbf{u} \in B([0,+\infty), X_\alpha) \cap C^{\alpha/2}([0,+\infty); X_0)$ and

$$(1+t)\sup_{t\ge1}\|D_t\mathbf{u}(t,\cdot)\|_{X_\alpha} + \sup_{1\le s<t}(1+s)\frac{\|D_t\mathbf{u}(t,\cdot)-D_t\mathbf{u}(s,\cdot)\|_{X_0}}{(t-s)^{\alpha/2}} \le C_{10}[[\psi]]_4, \qquad \forall t \ge 1. \tag{4.60}$$

Since $\mathbf{u} + \mathcal{N}\psi = D_t\mathbf{w} \in B([0,+\infty), D_L(1+\alpha/2,\infty))$ (see (4.56)) and $\mathcal{N}\psi \in B([0,+\infty), X_{2+\alpha})$ (see (4.53)), we easily deduce that $\mathbf{u} \in B([0,+\infty), X_{2+\alpha})$ and there exists a positive constant C_{11} such that

$$\sup_{t\ge1}(1+t)\|\mathbf{u}(t,\cdot)\|_{X_{2+\alpha}} \le C_{11}[[\psi]]_4. \tag{4.61}$$

Now, taking (4.56), (4.60), (4.61) and applying Lemma 3.4 to the function $\mathbf{v}$ defined by $\mathbf{v}(t,\cdot) = \mathbf{u}(t+1,\cdot)$ for any $t \geq 0$, we easily get (4.52). The proof is now complete. □

We can now prove Theorem 4.7.

Proof of Theorem 4.7. The proof follows from Theorem 4.6 and Lemmas 4.8, 4.9 and A.3. Indeed, our assumptions on the triplet $(\mathbf{u}_0, \mathbf{g}, \psi)$, the compatibility conditions in (4.36) and Theorem 4.6 imply that problem (4.18ii) admits a unique solution $\mathbf{w}$ which belongs to $\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)$ for any $T > 0$ and satisfies (4.33).

Let us recall that the solution to problem (4.18ii) can be split as $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where

$$\mathbf{w}_1(t,\cdot) := e^{tL}\mathbf{u}_0 + \int_0^t e^{(t-s)L}\big(\mathbf{g}(s,\cdot) + \psi(s,\cdot)\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi(s,\cdot)\big)ds,$$

$$\mathbf{w}_2(t,\cdot) := -L\int_0^t e^{(t-s)L}\mathcal{N}\psi(s,\cdot)ds, \qquad \forall t \geq 0.$$

Throughout the rest of the proof, we denote by C_j ($j \in \mathbb{N}$) positive constants, *independent* of the data and t.

Due to Lemma 4.5iii and (4.53), the function $t \mapsto \mathbf{g}(t,\cdot) + \psi(t,\cdot)\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi(t,\cdot)$ belongs to $(I-P)(X_0)$ for any $t \geq 0$. Moreover, it satisfies condition (4.38). To check it, we begin by observing that

$$\sup_{0\leq s<t}(1+s)\frac{\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{\alpha/2}} < 2[[\psi]]_2 \tag{4.62}$$

and this follows easily observing that

$$\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C(\mathbb{R})} \leq (1+s)^{-1}[[\psi]]_2(t-s)^{(1+\alpha)/2} \leq (1+s)^{-1}[[\psi]]_2(t-s)^{\alpha/2},$$

if $0 \leq s < t$ and $t \leq s+1$, while

$$\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C(\mathbb{R})} \leq \big(\|\psi(t,\cdot)\|_{C(\mathbb{R})} + \|\psi(s,\cdot)\|_{C(\mathbb{R})}\big)(t-s)^{\alpha/2} \leq 2(1+s)^{-1}[[\psi]]_2(t-s)^{\alpha/2},$$

if $0 \leq s < t$ and $t > s+1$.

Now, taking (4.35), (4.62) and Lemma 4.5 into account, we easily deduce

$$\|\mathcal{N}\psi(t,\cdot)-\mathcal{N}\psi(s,\cdot)\|_{X_2} \leq C_1\|\psi(t,\cdot)-\psi(s,\cdot)\|_{C^1(\mathbb{R})} \leq 2C_1[[\psi]]_2(1+s)^{-1}(t-s)^{\alpha/2}, \quad \forall 0 \leq s < t. \tag{4.63}$$

Hence, from (4.53), (4.62) and (4.63) we get

$$[[\psi\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi]]_3 \leq C_2[[\psi]]_2. \tag{4.64}$$

Applying Lemma 4.8 with $X = (I-P)(X_0)$, $A = L_{|(I-P)}$, $f = \mathbf{g} + \psi\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi$, $u_0 = \mathbf{w}_0$, and taking into account the characterization of the interpolation space $D_L(\alpha/2,\infty)$ given in Theorem 4.1, and (4.64), we get

$$\begin{aligned}&\sup_{t\geq 1}(1+t)\big(\|\mathbf{w}_1(t,\cdot)\|_{D(L)} + \|D_t\mathbf{w}_1(t,\cdot)\|_{X_\alpha}\big) + \sup_{1\leq s<t}(1+s)\frac{\|D_t\mathbf{w}_1(t,\cdot)-D_t\mathbf{w}_1(s,\cdot)\|_{X_0}}{(t-s)^{\alpha/2}}\\ &\leq C_3\big(\|\mathbf{w}_0\|_{X_0} + [[\mathbf{g}]]_1 + [[\psi]]_2\big).\end{aligned} \tag{4.65}$$

Since $D_t\mathbf{w}_1 = L\mathbf{w}_1 + \mathbf{g} + \psi\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi$ and $\mathbf{g} + \psi\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi \in B([0,+\infty);X_\alpha)$ with

$$\sup_{t\geq 0}(1+t)\|\mathbf{g}(t,\cdot) + \psi(t,\cdot)\mathbf{U}_0 + \mathcal{L}\mathcal{N}\psi(t,\cdot)\|_{X_\alpha} \leq C_4\big([[\mathbf{g}]]_1 + [[\psi]]_2\big),$$

we easily deduce that

$$\sup_{t\geq 1}(1+t)\|L\mathbf{w}_1(t,\cdot)\|_{X_\alpha} \leq C_5\big(\|\mathbf{w}_0\|_{X_0} + [[\mathbf{g}]]_1 + [[\psi]]_2\big). \tag{4.66}$$

From the Schauder estimate in [1] and from (4.65) and (4.66) it easily follows that $\mathbf{w}_1$ belongs to $B([0,+\infty);X_{2+\alpha})$ and

$$\sup_{t\geq 1}(1+t)\|\mathbf{w}_1(t,\cdot)\|_{X_{2+\alpha}} \leq C_6\big(\|\mathbf{w}_0\|_{X_0} + [[\mathbf{g}]]_1 + [[\psi]]_2\big). \tag{4.67}$$

Now, applying Lemma 3.4 to the function $\mathbf{u}(t,\cdot) = \mathbf{w}(t+1,\cdot)$ and taking (4.65) and (4.67) into account, we get

$$\sup_{1\leq s<t}(1+s)\frac{\|\mathbf{w}_1(t,\cdot)-\mathbf{w}_1(s,\cdot)\|_{X_2}}{(t-s)^{\alpha/2}} + \sup_{1\leq s<t}(1+s)\frac{\|\mathbf{w}_1(t,\cdot)-\mathbf{w}_1(s,\cdot)\|_{X_1}}{(t-s)^{(1+\alpha)/2}} \leq C_7\big(\|\mathbf{w}_0\|_{X_0} + [[\mathbf{g}]]_1 + [[\psi]]_2\big). \tag{4.68}$$

As far as the term $\mathbf{w}_2$ is concerned, we observe that the assumptions of Lemma 4.9 are satisfied by ψ. Hence, from (4.33) (with $T=1$), (4.52) (applied to the function $\mathbf{w}_2$), (4.65), (4.67), (4.68) and Lemma A.3, we deduce that $\mathbf{w}$ belongs to $\mathcal{X}_{\mathbf{w}}$ and satisfies (4.37). □

4.4 Stability results

This subsection is devoted to proving that the null solution to problem (2.15) is stable with respect to smooth and sufficiently small perturbations. Theorem 4.7 provides us an useful tool in order to prove our stability result. As it has already been pointed out, we can limit ourselves to dealing with problems (4.17i) and (4.17ii) where $\widetilde{\mathcal{H}}$, $\mathcal{K}$ and $\mathcal{G}$ are given, respectively, by (4.13), (4.14)-(4.16) and (2.14).

We solve system (4.17) by a fixed-point argument. Indeed, any sufficiently smooth solution to system (4.17) is a fixed-point of the operator $\Gamma(q,\mathbf{w}) = (\Gamma_1(q,\mathbf{w}),\Gamma_2(q,\mathbf{w}))$ defined by

$$\begin{aligned}
\Gamma_1(q,\mathbf{w})(t,\cdot) &= T(t)\big(\exp(M(\mathbf{u}_0)/2)-1\big) + \int_0^t T(t-s)\widetilde{\mathcal{H}}(q(s,\cdot),\mathbf{w}(s,\cdot))ds, \qquad t\geq 0,\\
\Gamma_2(q,\mathbf{w})(t,\cdot) &= e^{tL}(I-P)\mathbf{u}_0 + \int_0^t e^{(t-s)L}\big(\mathcal{K}(q(s,\cdot),\mathbf{w}(s,\cdot)) + \mathcal{L}\mathcal{N}\widetilde{g}(q(s,\cdot),\mathbf{w}(s,\cdot)\big)ds\\
&\quad - L\int_0^t e^{(t-s)L}\mathcal{N}\widetilde{g}(q(s,\cdot),\mathbf{w}(s,\cdot))ds, \qquad t\geq 0.
\end{aligned} \tag{4.69}$$

Let us introduce the Banach space where we are going to solve the fixed-point equation for $(q,\mathbf{w})$. For this purpose, we denote by $\mathcal{X}_{q,\mathbf{w}}$ the Banach space of all the pairs $(q,\mathbf{w})\in\mathcal{X}_q\times\mathcal{X}_{\mathbf{w}}$ such that $\mathbf{w}(t,\cdot)\in(I-P)(X_0)$ for any $t\geq 0$ (see Definition 3.3), endowed with the norm $\|(q,\mathbf{w})\|_{\mathcal{X}_{q,\mathbf{w}}} = \|q\|_{\mathcal{X}_q} + \|\mathbf{w}\|_{\mathcal{X}_{\mathbf{w}}}$, and we denote by $B(\mathbf{0},\rho)$ its closed ball with center at $(0,\mathbf{0})$ and radius ρ.

The main result of this subsection is the following theorem.

Theorem 4.10. *Suppose that $q_0\in C^{2+\alpha}(\mathbb{R})$, $\mathbf{w}_0\in(I-P)(X_{2+\alpha})$ satisfy the compatibility conditions*

$$\mathcal{B}\mathbf{w}_0 = \mathcal{G}\left(\mathbf{w}_0 + 2\log(q_0+1)\mathbf{U}_0\right), \qquad B_0\mathcal{L}(\mathbf{w}_0+\mathcal{K}(q_0,\mathbf{w}_0)) = 0. \tag{4.70}$$

Then, there are positive constants ρ_0 and ρ such that, if

$$\|q_0\|_{C^{2+\alpha}(\mathbb{R})} + \|\mathbf{w}_0\|_{X_{2+\alpha}} \leq \rho_0,$$

problem (4.17) admits a unique solution $(q,\mathbf{w})\in B(\mathbf{0},\rho)$ satisfying $(q(0,\cdot),\mathbf{w}_0(0,\cdot)) = (q_0,\mathbf{w}_0)$. Moreover, there exists a positive constant C, independent of $(q,\mathbf{w})$, such that

$$\begin{aligned}
&\|(q,\mathbf{w})\|_{\mathcal{X}_{q,\mathbf{w}}} + \sup_{0\leq s<t}(1+s)\left(\frac{\|q_y(t,\cdot)-q_t(s,\cdot)\|_{C(\mathbb{R})}}{(t-s)^{(1+\alpha)/2}} + \frac{\|\mathbf{w}(t,\cdot)-\mathbf{w}(s,\cdot)\|_{X_1}}{(t-s)^{(1+\alpha)/2}}\right)\\
&\leq C\big(\|q_0\|_{C^{2+\alpha}(\mathbb{R})} + \|\mathbf{w}_0\|_{X_{2+\alpha}}\big).
\end{aligned} \tag{4.71}$$

Proof. Let us prove that the operator Γ defined by (4.69) is a contraction mapping in the space

$$B(\mathbf{0},(q_0,\mathbf{w}_0),\rho) := \{(q,\mathbf{w}) \in B(\mathbf{0},\rho) : (q(0,\cdot),\mathbf{w}(0,\cdot)) = (q_0,\mathbf{w}_0)\},$$

provided ρ_0 and ρ are sufficiently small. For this purpose let us estimate the function $\widetilde{g}$, $\widetilde{\mathcal{H}}$ and $\mathcal{K}$. We begin with the function $\widetilde{g}$ (see (4.15)). Let us observe that, taking Lemma 3.4 into account, one can easily show that there exists a positive constant C, *independent* of q, such that

$$\sup_{s\geq 0}(1+s)^{j/2}\left(\|D_y^j q(s,\cdot)\|_{C^\alpha(\mathbb{R})} + \|D_y^j q\|_{C^{(i+\alpha)/2}([s,+\infty);C(\mathbb{R}))}\right) \leq C\|q\|_{\mathcal{X}_q}, \qquad \forall 0 \leq i,j \leq 1.$$

Since $\widetilde{g}$ is a product of functions belonging to $C^{(1+\alpha)/2,1+\alpha}([0,+\infty)\times\mathbb{R})$, we easily deduce that, if ρ is taken sufficiently small so that both $1+q$ and $1-v(0)+v_x(0)$ do never vanish for $(q,\mathbf{w}) \in B(\mathbf{0},\rho)$ (e.g. $\rho \leq 1/2$), then $\widetilde{g}(q,\mathbf{w}) \in C^{(1+\alpha)/2,1+\alpha}([0,+\infty)\times\mathbb{R})$. Moreover, long but straightforward computations and the fact that $\widetilde{g}$ is quadratic near $\mathbf{0}$ show that there exists a positive and continuous function K_1 vanishing at 0 such that

$$[[\widetilde{g}(q_2,\mathbf{w}_2) - \widetilde{g}(q_1,\mathbf{w}_1)]]_2 \leq K_1(\rho)\|(q_2,\mathbf{w}_2) - (q_1,\mathbf{w}_1)\|_{\mathcal{X}_{q,\mathbf{w}}}, \tag{4.72}$$

for any $(q_j,\mathbf{w}_j) \in B(\mathbf{0},\rho)$, $(j=1,2)$.

We now consider the operator $\widetilde{\mathcal{H}}$ (see (4.13)). All the terms in the definition of $\widetilde{\mathcal{H}}$ belong to $C([0,+\infty);C^\alpha(\mathbb{R})) \cap C^{\alpha/2}([0,+\infty);C(\mathbb{R}))$ for any $(q,\mathbf{w}) \in B(\mathbf{0},\rho)$ for a sufficiently small ρ, since they are products of functions belonging to such spaces. Moreover,

$$\begin{aligned}&\sup_{t\geq 0}(1+t)^{3/2}\|\widetilde{\mathcal{H}}(q_2(t,\cdot),\mathbf{w}_2(t,\cdot)) - \widetilde{\mathcal{H}}(q_1(t,\cdot),\mathbf{w}_1(t,\cdot))\|_{C^\alpha(\mathbb{R})}\\ +\ &\sup_{s\geq 0}(1+s)^{3/2}[\widetilde{\mathcal{H}}(q_2,\mathbf{w}_2) - \widetilde{\mathcal{H}}(q_1,\mathbf{w}_1)]_{C^{\alpha/2}([s,+\infty);C(\mathbb{R}))} \leq K_2(\rho)\|(q_2,\mathbf{w}_2) - (q_1,\mathbf{w}_1)\|_{\mathcal{X}_{q,\mathbf{w}}},\end{aligned}$$

for any $(q_j,\mathbf{w}_j) \in B(\mathbf{0},\rho)$ $(j=1,2)$ and some positive and continuous function K_2 vanishing at 0. In particular, we get

$$[[\widetilde{\mathcal{H}}(q_2,\mathbf{w}_2) - \widetilde{\mathcal{H}}(q_1,\mathbf{w}_1)]]_0 \leq K_2(\rho)\|(q_2,\mathbf{w}_2) - (q_1,\mathbf{w}_1)\|_{\mathcal{X}_{q,\mathbf{w}}}, \tag{4.73}$$

(see (4.19)).

Similarly, for any $(q,\mathbf{w}) \in B(0,\rho)$ the function $\mathcal{K}(q,\mathbf{w})$ defined by the right-hand side of (4.16) belongs to $C^{\alpha/2}([0,+\infty);C(\mathbb{R})) \cap C([0,+\infty);C^\alpha(\mathbb{R}))$. Moreover,

$$[[\mathcal{K}(q_2,\mathbf{w}_2) - \mathcal{K}(q_1,\mathbf{w}_1)]]_1 \leq K_3(\rho)\|(q_2,\mathbf{w}_2) - (q_1,\mathbf{w}_1)\|_{\mathcal{X}_{q,\mathbf{w}}}, \tag{4.74}$$

for any $(q_j,\mathbf{w}_j) \in B(\mathbf{0},\rho)$ $(j=1,2)$ (see (4.34)) and some positive and continuous function K_3 going to 0 as ρ tends to 0.

Let us now observe that, since $\widetilde{g}(0,\mathbf{0}) = 0$, $\widetilde{\mathcal{H}}(0,\mathbf{0}) = \widetilde{\mathcal{K}}(0,\mathbf{0}) = \mathbf{0}$, from (4.72)–(4.74) we deduce

$$[[\widetilde{\mathcal{H}}(q,\mathbf{w})]]_0 + [[\widetilde{\mathcal{K}}(q,\mathbf{w})]]_1 + [[\widetilde{g}(q,\mathbf{w})]]_2 \leq \big(K_1(\rho)+K_2(\rho)+K_3(\rho)\big)\|(q,\mathbf{w})\|_{\mathcal{X}_{q,\mathbf{w}}}, \tag{4.75}$$

for any $(q,\mathbf{w}) \in B(\mathbf{0},\rho)$.

Now, taking Theorems 4.4, 4.7 and all the above estimates into account, we easily deduce that, if ρ is sufficiently small, then $\Gamma(q,\mathbf{w}) \in \mathcal{X}_{q,\mathbf{w}}$ for any $(q,\mathbf{w}) \in B(\mathbf{0},(q_0,\mathbf{w}_0),\rho)$ (observe that the compatibility conditions in Theorem 4.6 are satisfied by virtue of (4.70), since $(q(0,\cdot),\mathbf{w}(0,\cdot)) = (q_0,\mathbf{w}_0)$).

Moreover, from (4.20), (4.37), (4.75) we immediately deduce that

$$\|\Gamma(q,\mathbf{w})\|_{\mathcal{X}_{q,\mathbf{w}}} \leq C\big(\|q_0\|_{C^{2+\alpha}(\mathbb{R})} + \|\mathbf{w}_0\|_{X_{2+\alpha}} + K_4(\rho)\|(q,\mathbf{w})\|_{\mathcal{X}_{q,\mathbf{w}}}\big),$$

for some constant C, *independent* of $(q_0,\mathbf{w}_0)$ and some positive and continuous function K_4 vanishing at zero.

Similarly, since, for any $(q_1,\mathbf{w}_1),(q_2,\mathbf{w}_2)\in B(\mathbf{0},(q_0,\mathbf{w}_0),\rho)$ the function $\Gamma(q_2,\mathbf{w}_2)-\Gamma(q_1,\mathbf{w}_1)$ turns out to be a solution to system (4.17) (with $(\widetilde{\mathcal{H}}(q,\mathbf{w}),\widetilde{\mathcal{K}}(q,\mathbf{w}),\widetilde{g}(q,\mathbf{w}),q_0,\mathbf{w}_0)$ replaced by $(\widetilde{\mathcal{H}}(q_2,\mathbf{w}_2)-\widetilde{\mathcal{H}}(q_1,\mathbf{w}_1),\widetilde{\mathcal{K}}(q_2,\mathbf{w}_2)-\widetilde{\mathcal{K}}(q_1,\mathbf{w}_1),\widetilde{g}(q_2,\mathbf{w}_2)-\widetilde{g}(q_1,\mathbf{w}_1),0,\mathbf{0})$), from (4.20), (4.37), (4.72)–(4.74), we deduce

$$\|\Gamma(q_2,\mathbf{w}_2)-\Gamma(q_1,\mathbf{w}_1)\|_{\mathcal{X}_{q,\mathbf{w}}}\le K_5(\rho)\|(q_2,\mathbf{w}_2)-(q_1,\mathbf{w}_1)\|_{\mathcal{X}_{q,\mathbf{w}}},$$

for some positive and continuous function K_5 vanishing at zero.

Choose now a pair (ρ,ρ_0) solution to the system of inequalities

$$\begin{cases} C(\rho_0+K_4(\rho)\rho)\le\rho,\\ K_5(\rho)\le\frac{1}{2},\\ 0<\rho_0\le\rho\le\frac{1}{2}. \end{cases}$$

Then Γ turns out to be a 1/2-contraction mapping in $B(\mathbf{0},(q_0,\mathbf{w}_0),\rho)$ and, consequently, the fixed-point equation $\Gamma(q,\mathbf{w})=(q,\mathbf{w})$ admits a unique solution $(q,\mathbf{w})\in B(\mathbf{0},(q_0,\mathbf{w}_0),\rho)$ solving system (4.17). Estimate (4.71) now follows easily. □

5 Concluding remarks

The results in Section 4 imply that there exist two positive constants ρ_0 and ρ such that if $\mathbf{u}_0$ belongs to $B(0,\rho_0)\subset X_{2+\alpha}$ and satisfies the compatibility conditions (4.32), then the initial value problem $\mathbf{u}(0,\cdot)=\mathbf{u}_0$ for problem (2.15) admits a unique globally defined solution $\mathbf{u}\in B(0,\rho)\subset\mathcal{X}_{1+\alpha/2,2+\alpha}(0,\infty)$. Moreover, there exists a positive constant C such that

$$(1+t)^{\alpha_1+\alpha_3/2}\left(\|D_t^{\alpha_1}D_x^{\alpha_2}D_y^{\alpha_3}\mathbf{u}\|_{C^{\alpha/2}([t,+\infty);X_0)}+\|D_t^{\alpha_1}D_x^{\alpha_2}D_y^{\alpha_3}\mathbf{u}(t,\cdot)\|_{X_0}\right)\le C\|\mathbf{u}_0\|_{X_{2+\alpha}},$$

for any $t\ge0$ and any $2\alpha_1+\alpha_2+\alpha_3\le2$.

One can show that, if ρ_0 is sufficiently small, then $\mathbf{u}$ is, actually, the *unique* solution in $\mathcal{X}_{1+\alpha/2,2+\alpha}(0,\infty)$ to the initial value problem $\mathbf{u}(0,\cdot)=\mathbf{u}_0$ for problem (2.15). To do this, the main step is to show that for any $t_0>0$ and any small $\mathbf{u}_0\in X_{2+\alpha}$ satisfying the due compatibility conditions at $t=t_0$, the problem

$$\begin{cases} D_t\mathbf{u}(t,\cdot)=\mathcal{L}\mathbf{u}(t,\cdot)+\mathcal{F}(\mathbf{u}(t,\cdot)), & t\in[t_0,t_0+\delta],\\ \mathcal{B}\mathbf{u}(t,\cdot)=\mathcal{G}(\mathbf{u}(t,\cdot)), & t\in[t_0,t_0+\delta],\\ \mathbf{u}(t_0,\cdot)=\mathbf{u}_0, \end{cases}$$

is uniquely solvable in a large ball of $\mathcal{X}_{1+\alpha/2,2+\alpha}(t_0,t_0+\delta)$ for some small $\delta>0$ (independent of t_0). See [7, Theorem 4.1] for more details.

Coming back to problem (2.1)-(2.4) the previous results ensure that the planar TW is stable with respect to small and sufficiently smooth perturbations. In particular, the perturbed front ϕ stays bounded and close to the front $-t$ corresponding to the TW. Moreover, its derivatives decrease polynomially to zero. To be more precise, there exists a positive constant C such that

$$\|D_t^{\alpha_1}D_y^{\alpha_2}\big(\phi(t,\cdot)+t\big)\|_{C^\alpha(\mathbb{R})}\le\frac{C}{(1+t)^{\alpha_1+\alpha_2/2}},$$

for any $t\ge0$ and any $2\alpha_1+\alpha_2\le2$. Moreover, the functions Θ^1 and S^1 stay close to Θ^0 and 0, respectively, and

$$\begin{aligned}&\|D_t^{\alpha_1}D_x^{\alpha_2}D_y^{\alpha_3}\big(\Theta^1(t,\cdot)-\Theta^0\big)\|_{C^\alpha(\mathbb{R}^2_-)}+\|D_t^{\alpha_1}D_x^{\alpha_2}D_y^{\alpha_3}S^1(t,\cdot)\|_{C^\alpha(\mathbb{R}^2_-)}+\|D_t^{\alpha_1}D_x^{\alpha_2}D_y^{\alpha_3}S^1(t,\cdot)\|_{C^\alpha(\mathbb{R}^2_+)}\\ \le\ &\frac{C}{(1+t)^{\alpha_1+\alpha_3/2}},\end{aligned}$$

for any $2\alpha_1+\alpha_2+\alpha_3\le2$, where $\mathbb{R}^2_+:=\mathbb{R}_+\times\mathbb{R}$.

A Additional tools

We begin this subsection with an abstract lemma which plays a crucial role in the proof of Theorem 4.1.

Lemma A.1. *Let X be a Banach space, $A : D(A) \subset X \to X$, $B : D(B) \subset X \to X$ be two closed operators with non empty resolvent sets. Then A and B commute in the resolvent sense if and only if*

$$R(\omega, B)(D(A)) \subset D(A) \text{ and } AR(\omega, B)x = R(\omega, B)Ax, \qquad \forall \omega \in \rho(B), \quad \forall x \in D(A). \tag{A.1}$$

Proof. Let us assume that A and B commute in the resolvent sense and prove that condition (A.1) is satisfied. Fix $x \in D(A)$, $\sigma \in \rho(A)$ and let $y \in X$ such that $x = R(\sigma, A)y$. Then, for any $\omega \in \rho(B)$

$$\begin{aligned} R(\omega, B)Ax &= R(\omega, B)AR(\sigma, A)y = R(\omega, B)(A - \sigma I + \sigma I)R(\sigma, A)y \\ &= -R(\omega, B)y + \sigma R(\omega, B)R(\sigma, A)y = -R(\omega, B)y + \sigma R(\sigma, A)R(\omega, B)y \\ &= (\sigma R(\sigma, A) - I)R(\omega, B)y = AR(\sigma, A)R(\omega, B)y = AR(\omega, B)R(\sigma, A)y \\ &= AR(\omega, B)x, \end{aligned}$$

so that (A.1) is satisfied.

Conversely, let us prove that, if (A.1) holds, then A and B commute in the resolvent sense. Fix $y \in X$ and take $\sigma \in \rho(A)$, $\omega \in \rho(B)$. Applying (A.1) to $x = R(\sigma, A)y$ we get

$$\begin{aligned} AR(\omega, B)R(\sigma, A)y &= R(\omega, B)AR(\sigma, A)y = R(\omega, B)(A - \sigma I + \sigma I)R(\sigma, A)y \\ &= -R(\omega, B)y + \sigma R(\omega, B)R(\sigma, A)y. \end{aligned}$$

Hence, $(A - \sigma I)R(\omega, B)R(\sigma, A)y = -R(\omega, B)y$. Applying $R(\sigma, A)$ to both sides of the previous equation we get $R(\omega, B)R(\sigma, A)y = R(\sigma, A)R(\omega, B)y$ so that A and B commute in the resolvent sense. □

The technique we used, in the case $\lambda = 0$, to transform problem (1.1) into an equivalent one, which is somewhat simpler to be studied, works as well in the case of problem (1.1) with $\lambda \neq 0$. We still get a problem similar to (2.15), where now the operators $\mathcal{L}$ and $\mathcal{B}$ have to be replaced by the operators

$$\mathcal{L}\mathbf{u} = (\Delta v - v_x, \Delta w - w_x - \lambda \Delta v, \Delta h + h_x), \qquad \mathcal{B}\mathbf{u} = \begin{pmatrix} \lambda v(0,\cdot) - w(0,\cdot) + h(0,\cdot) \\ \lambda v(0,\cdot) + \lambda v_x(0,\cdot) - w_x(0,\cdot) - h_x(0,\cdot) \\ v(0,\cdot) + h(0,\cdot) - v_x(0,\cdot) \end{pmatrix},$$

where, as usual, we set $\mathbf{u} = (v, w, h)$.

The pair $(\mathcal{L}, \mathcal{B})$ generates an analytic semigroup in X_0 for any $\lambda \in \mathbb{R}$ with domain still given by (4.3) (see [7, Theorem A.2]). We still can decouple $\mathcal{L}$ into the sum of the two operators L_1 and L_2 defined by

$$L_1\mathbf{u} = (v_{xx} - v_x, w_{xx} - w_x - \lambda v_{xx}, h_{xx} + h_x), \qquad L_2\mathbf{u} = (v_{yy}, w_{yy} - \lambda v_{yy}, h_{yy}),$$

with domains

$$D(L_1) = \{\mathbf{u} : D_x^j \mathbf{u} \in X_0 \text{ for } j = 0, 1, 2, \ B\mathbf{u} = 0\}, \qquad D(L_2) = \{\mathbf{u} : D_y^j \mathbf{u} \in X_0 \text{ for } j = 0, 1, 2\}.$$

Although these operators are generators of analytic semigroups in X_0, they do not commute in the resolvent sense due to the coupling between v and w induced by the parameter λ.

Theorem A.2. *Suppose that $\lambda \neq 0$. Then, the operators $L_1 : D(L_1) \subset X_0 \to X_0$ and $L_2 : D(L_2) \subset X_0 \to X_0$ are generators of analytic semigroups in X_0 not commuting in the resolvent sense.*

Proof. Showing that $L_j : D(L_j) \subset X_0 \to X_0$ $(j = 1, 2)$ generates an analytic semigroup in X_0 is an easy exercise and, hence, it is left to the reader. In particular, straightforward computations show that for

any $\omega \in \rho(L)$ and any $\lambda \in \mathbb{R}$ it holds that

$$\begin{aligned}
[R(\omega, L_1)\mathbf{f}]_1(x,y) &= \frac{1}{\sqrt{1+4\omega}}\left(\int_{-\infty}^0 e^{-\mu_1 t} f(t+x,y)dt + \int_x^0 e^{\mu_2 t} f(x-t,y)dt\right) \\
&\quad + a_1 e^x \int_{-\infty}^0 e^{-\mu_1(t+x)} g(t,y)dt + a_1 \int_{-\infty}^0 e^{\mu_2(t+x)} k(t,y)dt \\
&\quad + a_2 e^x \int_{-\infty}^0 t e^{-\mu_1(t+x)} f(t,y)dt + a_3 e^x \int_{-\infty}^0 e^{-\mu_1(t+x)} f(t,y)dt,
\end{aligned}$$

$$\begin{aligned}
[R(\omega, L_1)\mathbf{f}]_2(x,y) &= \frac{1}{\sqrt{1+4\omega}} \int_{-\infty}^0 e^{-\mu_1 t}\big(g - \lambda D_{xx}[R(\omega, L_1)\mathbf{f}]_1\big)(t+x,y)dt \\
&\quad + \frac{1}{\sqrt{1+4\omega}} \int_x^0 e^{\mu_2 t}\big(g - \lambda D_{xx}[R(\omega, L_1)\mathbf{f}]_1\big)(x-t,y)dt \\
&\quad + a_4 e^x \int_{-\infty}^0 e^{-\mu_1(t+x)} g(t,y)dt + a_5 \int_{-\infty}^0 e^{\mu_2(t+x)} k(t,y)dt \\
&\quad + a_6 e^x \int_{-\infty}^0 t e^{-\mu_1(t+x)} f(t,y)dt + a_7 e^x \int_{-\infty}^0 e^{-\mu_1(t+x)} f(t,y)dt,
\end{aligned}$$

$$\begin{aligned}
[R(\omega, L_1)\mathbf{f}]_3(x,y) &= \frac{1}{\sqrt{1+4\omega}}\left(\int_x^0 e^{-\mu_1 t} k(x-t,y)dt + \int_{-\infty}^0 e^{\mu_2 t} k(t+x,y)dt\right) \\
&\quad + a_8 \int_{-\infty}^0 e^{-\mu_1(t+x)} f(t,y)dt + a_9 \int_{-\infty}^0 t e^{-\mu_1(t+x)} f(t,y)dt \\
&\quad + a_{10} \int_{-\infty}^0 e^{-\mu_1(t+x)} g(t,y)dt + a_{11} \int_{-\infty}^0 e^{t-\mu_1(t+x)} k(t,y)dt,
\end{aligned}$$

for any $(x,y) \in \mathbb{R}^2_-$ and suitable coefficients $a_j \neq 0$ $(j = 1, \dots, 11)$ depending on λ, where $\mu_j = \frac{1}{2} + (-1)\frac{1}{2}\sqrt{1+4\omega}$ for $j = 1, 2$, and $\mathbf{f} = (f, g, k)$.

An immediate computation shows that

$$R(\omega, L_1)L_2\mathbf{f} = R(\omega, L_1)\big(\mathbf{f}_{yy} - (0, \lambda f_{yy}, 0)\big), \qquad L_2 R(\omega, L_1)\mathbf{f} = R(\omega, L_1)\mathbf{f}_{yy} - \lambda(0, D_{yy}[R(\omega, L_1)\mathbf{f}]_1, 0),$$

for any $\mathbf{f} \in D(L_2)$. Hence, $R(\omega, L_1)L_2\mathbf{f} = L_2 R(\omega, L_1)\mathbf{f}$ if and only if

$$R(\omega, L_1)(0, f_{yy}, 0) = (0, D_{yy}[R(\omega, L_1)\mathbf{f}]_1, 0), \tag{A.2}$$

Take $\mathbf{f}(x,y) = (0, e^x \widetilde{g}(y), 0)$ for some smooth function g such that $\widetilde{g}_{yy}$ does not vanish identically. Obviously $\mathbf{f} \in D(L_2)$. Moreover, the derivative

$$D_{yy}[R(\omega, L_1)\mathbf{f}]_1(x,y) = a_1 \widetilde{g}''(y) \int_{-\infty}^0 e^{\mu_2(t+x)} dt, \qquad \forall (x,y) \in \overline{\mathbb{R}}^2_-,$$

does not vanish identically. Hence, (A.2) is not satisfied and consequently, by virtue of Lemma A.1, L_1 and L_2 do not commute in the resolvent sense. □

Lemma A.3. *Let X be a Banach space and let $f : [0, +\infty) \to X$ be such that $f \in C^\alpha([0,1]; X)$ and*

$$\sup_{1 \le s < t} s \, \frac{\|f(t) - f(s)\|_X}{(t-s)^\alpha} < +\infty, \tag{A.3}$$

for some $\alpha \in (0,1)$. Then, $f \in C^\alpha([0, +\infty); X)$ and

$$[f]_{C^\alpha([0,+\infty);X)} \le [f]_{C^\alpha([0,1];X)} + \sup_{1 \le s < t} s \, \frac{\|f(t) - f(s)\|_X}{(t-s)^\alpha}. \tag{A.4}$$

In particular,

$$\sup_{0\le s<t}(1+s)\frac{\|f(t)-f(s)\|_X}{(t-s)^\alpha}\le 2\left([f]_{C^\alpha([0,1];X)}+\sup_{1\le s<t}s\,\frac{\|f(t)-f(s)\|_X}{(t-s)^\alpha}\right). \tag{A.5}$$

Proof. Since f satisfies (A.3), then, in particular,

$$[f]_{C^\alpha([1,+\infty))}\le\sup_{1\le s<t}s\,\frac{\|f(t)-f(s)\|_X}{(t-s)^\alpha}.$$

Moreover, if $s<1<t$, then

$$\begin{aligned}\|f(t)-f(s)\|_X &\le \|f(t)-f(1)\|_X+\|f(1)-f(s)\|_X\\ &\le [f]_{C^\alpha([0,1];X)}(1-s)^\alpha+\sup_{1\le r_1<r_2}r_1\frac{\|f(r_2)-f(r_1)\|_X}{(r_2-r_1)^\alpha}(t-1)^\alpha\\ &\le \left([f]_{C^\alpha([0,1];X)}+\sup_{1\le r_1<r_2}r_1\frac{\|f(r_2)-f(r_1)\|_X}{(r_2-r_1)^\alpha}\right)(t-s)^\alpha\end{aligned}$$

and (A.4), (A.5) easily follow. □

References

[1] AGMON S., DOUGLIS A., NIRENBERG L., *Estimates near the boundary for solutions to elliptic partial differential equation satisfying general boundary conditions II*, Comm. Pure Appl. Math., 17 (1964), pp. 35–92.

[2] BRAUNER C.-M., HULSHOF J., LUNARDI A., *A critical case of stability in a free boundary problem*, J. Evol. Eqns. 1 (2001), pp. 85–113.

[3] BRAUNER C.-M., LUNARDI A., *Instabilities in a two-dimensional combustion model with free boundary*, Arch. Rational Mech. Anal. 154 (2000), pp. 157–182.

[4] DA PRATO G., GRISVARD P., *Sommes d'opérateurs linéaires et équations différentielles opérationelles*, J. Math. pures et appl. 54 (1975), pp. 305–387.

[5] KAPITULA T., *Multidimensional stability of planar travelling waves*, Trans. A.M.S. 39 (1997), pp. 257–269.

[6] KURAMOTO Y., *Chemical waves and chemical turbulence*, 'Synergetics' (Proc. Internat. Workshop, Garmisch-Partenkirchen, 1977) (1977), pp. 164–173.

[7] LORENZI L., *Regularity and analyticity in a two-dimensional combustion model*, Adv. Diff. Eqns. 7, n. 11 (2002), pp. 1343–1376.

[8] LORENZI L., LUNARDI A., *A critical case of stability in a two-dimensional combustion model*, Nonlinear Anal. T.M.A. (to appear).

[9] LUNARDI A., Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser Verlag, Basel (1995).

[10] SIVASHINSKY G.I., *On flame propagation under conditions of stoichiometry*, SIAM J. Appl. Math. 39 (1980), pp. 299–310.