ENERGY CONCENTRATION FOR 2-DIMENSIONAL RADIALLY SYMMETRIC EQUIVARIANT HARMONIC MAP HEAT FLOWS

MICHIEL BERTSCH *, REIN VAN DER HOUT †, AND JOSEPHUS HULSHOF‡

Abstract. We give a description of singularity formation in terms of energy quanta for 2-dimensional radially symmetric equivariant harmonic map heat flows. Adapting Struwe's energy method we first establish a finite bubble tree result with a discrete multiple of energy quanta disappearing in the singularity. We then use intersection-comparison arguments to show that the bubble tree consists of a single bubble only and that there is a well defined scale $R_{\mathrm{BHK}}(t) \downarrow 0$ in which the solution converges to the standard harmonic map[1].

Key words. Singularity formation, energy method, energy quanta, bubble tree, bubbling off, single bubble, intersection-comparison.

AMS subject classifications. 35Bxx, 35Kxx, 46N20, 46N60, 53B50, 92Cxx

1. Introduction.

We consider solutions of the equation

$$u_t = \frac{\partial u}{\partial t} = \Delta u + u|\nabla u|^2, \tag{1.1}$$

where $u(\cdot, t) : D^2 \to S^2$ maps the unit disk in $\mathbb{R}^2$ to the unit sphere in $\mathbb{R}^3$. Here

$$|\nabla u|^2 = \sum_{j=1,2,3} \left(\left(\frac{\partial u_j}{\partial x} \right)^2 + \left(\frac{\partial u_j}{\partial y} \right)^2 \right).$$

It is well known ([8]) that, for given Dirichlet boundary conditions, smooth solutions do not always exist for all times. In the much more general context of harmonic map heat flow on a 2-dimensional Riemannian manifold, Struwe ([15]) has introduced an energy approach to show that singularity formation is not described by the natural parabolic selfsimilar scaling: if one zooms in along appropriate sequences, one identifies the spatial profile of a steady state, i.e. a harmonic map. In [14] and [10] it was in fact shown that, in general, a bubble tree develops, the bubbles consisting of harmonic maps with well defined energy quanta, the sum of which is precisely equal to the energy lost in the singularity. For a recent list of references we refer to [12].

In the present paper we restrict ourselves to radially symmetric equivariant solutions, which we recall and discuss in Section 1.1. It was shown in [19] that then a bubble tree consists of at most one single bubble. Our purpose here is to show that only one scale is needed to describe the singularity formation: if the solution becomes singular at a finite time $t = T$, then there is a *decreasing* function $R(t) \to 0$ as $t \uparrow T$, such that in terms of $\frac{r}{R(t)}$ the solution converges to a nonconstant radially symmetric harmonic map, which is unique modulo scaling and symmetry. It is the scale which Van den Berg, Hulshof and King use in their formal analysis in [17], and we shall indicate it by $R_{\mathrm{BHK}}(t)$.

The result will be presented in a completely self-contained form, with a simplified proof of the single bubble result. In a future paper we shall show that our methods can

*IAC-CNR and Tor Vergata University, Rome

†Department of Mathematics, VU-University Amsterdam

‡Department of Mathematics, VU-University Amsterdam

[1] We thank Sigurd Angenent, Jan Bouwe van den Berg, JF Williams, John King, Peter Topping and the referee

be adapted to establish similar results for general (not necessarily radially symmetric) equivariant solutions. This more general class is of particular interest since formal calculations in [18] suggest that, in this class, singularities do not occur generically.

We expect that our result will enable us to give a rigorous proof of the conjectured blow-up rates based on the formal asymptotics in [17], and in particular the "generic" decay rate of $R_{\mathrm{BHK}}(t)$:

$$R_{\mathrm{BHK}}(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2}.$$

This will also benefit the understanding of singularity formation in relation to non-uniqueness phenomena exhibited in, see [4, 16], the phenomenon of *reverse bubbling*. Although Freire [11] provided a seemingly natural uniqueness criterion for the heat flow of harmonic maps on Riemann surfaces, many delicate questions about non-uniqueness are still not settled. For flows on 3-dimensional domains uniqueness criteria are not known at all, see for example [3, 6, 9, 13] for nonuniqueness phenomena.

1.1. Radially symmetric equivariant solutions. Using polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$ these are of the form

$$u(x,y,t) = \begin{pmatrix} \sin\theta(r,t)\cos\phi \\ \sin\theta(r,t)\sin\phi \\ \cos\theta(r,t) \end{pmatrix}, \tag{1.2}$$

with $\theta(r,t)$ satisfying

$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - \frac{\sin 2\theta}{2r^2}, \tag{1.3}$$

or, in terms of $s = \log r$,

$$e^{2s}\theta_t = \theta_{ss} - \frac{\sin 2\theta}{2}. \tag{1.4}$$

Note that

$$|u_t|^2 = \theta_t^2, \quad |u_r|^2 = \theta_r^2, \quad |u_\phi|^2 = \sin^2\theta,$$

and that it makes good sense to reformulate the problem in terms of $\theta(r,t)$, where in principle $\theta(r,t)$ is not limited to the range $0 \le \theta \le \pi$. We note that equivariant solutions map the origin to one of the poles and that nontrivial radially symmetric equivariant equilibria of (1.1) having one of the poles in their image can all be written in the form $\theta = 2\arctan(qr)$ or $\pi - \theta = 2\arctan(qr)$. In rectangular coordinates these equilibria read

$$u_1 = \frac{2qx}{1+q^2(x^2+y^2)}, \; u_2 = \frac{2qy}{1+q^2(x^2+y^2)}, \; u_3 = \pm\frac{1-q^2(x^2+y^2)}{1+q^2(x^2+y^2)}, \tag{1.5}$$

and they are all smooth.

1.2. Singularity formation. For radially symmetric equivariant solutions we impose time-independent radially symmetric equivariant Dirichlet boundary conditions on the circle $r^2 = x^2 + y^2 = 1$, given by a single value θ_1 in the corresponding condition for θ at $r = 1$:

$$\theta(1,t) = \theta_1. \tag{1.6}$$

At $t = 0$ we impose smooth radially symmetric equivariant initial conditions equivalent to

$$\theta(r, 0) = \theta_0(r). \tag{1.7}$$

where we assume without loss of generality that the usual compatibility conditions in $r = 1$ are satisfied and that $\theta_0(0) = 0$.

By standard arguments there is (locally in time) a unique solution which is smooth for $t > 0$. Assuming for simplicity of the discussion below that $\theta_0(r)$ increases from 0 to θ_1 as r runs from 0 to 1, it depends on θ_1 how solutions will behave. If $0 < \theta_1 < \pi$, the initial values define a map $u_0 : D^2 \to S^2$ which covers only part of S^2, and the solution $\theta(r, t)$ is seen to exist as a smooth solution for all time $t > 0$, converging to

$$\Theta(r; q) = 2 \arctan(qr) \tag{1.8}$$

as $t \to \infty$, where $q = \tan \frac{\theta_1}{2}$. In particular $\theta(0, t) = 0$ for all $t > 0$, since writing $\theta(r, t) = rp(r, t)$, the function $p(r, t)$ solves

$$p_t = p_{rr} + \frac{3}{r} p_r + \frac{1}{r^2} \left(p - \frac{\sin 2rp}{2r} \right), \tag{1.9}$$

a semilinear parabolic equation in $\mathbb{R}^4$, in which the unknown function is radially symmetric and positive in $r = 0$. The (smooth) nonlinear term in (1.9) is incapable of causing blow-up, thanks to the equilibria (1.8) which provide supersolutions by taking q sufficiently large.

If $\pi < \theta_1 < 2\pi$, the initial values define a map $u_0 : D^2 \to S^2$ which covers part of S^2 twice, while there are no radially symmetric equivariant equilibria available satisfying the boundary condition $\theta(1, t) = \theta_1$ with $\theta = 0$ in $r = 0$. It was first suggested in [8] that consequently there must exist a minimal $0 < T < \infty$ such that as $t \uparrow T$, $\theta_r(0, t)$ blows up. In fact they showed this for a subsolution, which allowed them to establish that the part of the solution which runs from $\theta = 0$ to $\theta = \pi$ (as r runs from $r = 0$ to some $r = S(t) > 0$) disappears. In terms of the map $u(\cdot, t) : D^2 \to S^2$, a sphere bubbles off. For the remaining part of the solution the scenario is, in terms $\theta(r, t) - \pi$, basically the same as that for $\theta(r, t)$ in the case that $0 < \theta_1 < \pi$. The solution may be continued globally for $t > T$ with $\theta(0, t) = \pi$, converging to $\pi + 2 \arctan(qr)$, with $q = \tan \frac{\theta_1 - \pi}{2}$, or, alternatively, with $\theta(0, t) = 2\pi$, converging to $2\pi - 2 \arctan(q_1 r)$, with $q_1 = \tan \frac{2\pi - \theta_1}{2}$ if one uses the above mentioned reverse bubbling scenarios suggested by Topping. We emphasize that the assumption of radial symmetry forces the solution curve $r \to u(r, t)$, as long as it is smooth, to pass through the South Pole if it does so for $t = 0$. In the general (not necessarily radially symmetric) equivariant case this is no longer true.

This *jump* behaviour of radially symmetric solutions was studied in more detail in [17, 1, 2, 19], using matched asymptotic expansions and intersection-comparison techniques, but in the case of finite time blow-up, the peculiar nonselfsimilar blow-up rates calculated in [17] have not been proved yet.

In the radially symmetric equivariant setting with $\theta(0, t) = 0$ for $t < T$, several definitions of singularity formation as $t \uparrow T$ have been used:

(i) blow up of $\theta_r(0, t)$ at time T;

(ii) $S(t) \to 0$ as $t \to T$;

(iii) concentration of a positive amount of energy in the origin.

The equivalence of (i) and (iii) already follows from [15]. We clarify the equivalence between these three definitions, basing the analysis primarily on (iii), but in the end it will be (i) that defines the scale $R_{\mathrm{BHK}}(t)$ (see [17]) in which we conclude. This is one of the 3 scales that we use. The second scale is defined by $\theta(R(t),t)=\frac{\pi}{3}$ and the third scale is $R_{\frac{1}{2}}(t)$, the radius of the ball in which the total energy equals half an energy quantum. We recall that under the appropriate boundary conditions such as the ones we impose, the harmonic map heat flow is a gradient flow for the energy

$$\frac{1}{2}\int_{D^2}|\nabla u|^2, \tag{1.10}$$

and the energy drop in a singularity, i.e. the negative jump in (1.10), is known to be equal to the total energy 4π of the equilibria in (1.8) on the whole of $\mathbb{R}^2$, as follows by combining [14] and [19]. In Section 2 we outline a self-contained direct proof of the Struwe-Qing-Ding-Tian bubble tree result. Using the quite different arguments of intersection-comparison theory, we give a simple proof of the single bubble result in [19] and establish that $R_{\mathrm{BHK}}(t)$ is in fact a scale:

$$R_{\mathrm{BHK}}(t):=\frac{2}{\theta_r(0,t)}\downarrow 0 \quad \text{as } t\uparrow T.$$

In a future paper we shall concentrate on the single bubble tree issue for equivariant solutions, without the assumption of radial symmetry, refining the self-contained approach of the present paper.

2. Energy proof of the bubble tree result revisited. We write the energy as

$$E(r)=E(r;u)=E(r;\theta)=\frac{1}{2}\int_0^r\left(r\theta_r^2+\frac{1}{r}\sin^2\theta\right)dr, \tag{2.1}$$

in which we omit the factor 2π corresponding to integration with respect to ϕ. For $t<T$ and any $r_1\le 1$, the energy equality reads

$$E(r_1;\theta(\cdot,t))+\int_0^t\int_0^{r_1}\theta_t^2\,rdrdt=E(r_1;\theta_0)+r_1\int_0^t\theta_t(r,t)\theta_r(r,t)dt, \tag{2.2}$$

where the second term in the right hand side of (2.2) vanishes if $r_1=1$, implying

$$\int_0^T\int_0^1\theta_t^2\,rdrdt\le E(1;\theta_0), \tag{2.3}$$

and, for all $0<t<T$,

$$\int_0^1\left(r\theta_r^2+\frac{\sin^2\theta}{r}\right)dr\le 2E(1;\theta_0). \tag{2.4}$$

Note that the integral in (2.3) is invariant under parabolic scalings of r and t, and that the integral in (2.4) is invariant under scalings of r. We have

$$E(r;\Theta(\cdot;q))=1-\cos\Theta(r;q)=\frac{2q^2r^2}{1+q^2r^2}\to 2=E(\infty;\Theta(\cdot;q)), \tag{2.5}$$

as $r \to \infty$. The limit 2 is the energy quantum which will disappear in a singularity. It is useful to observe, see for example Lemma 20 in [5], that, for any smooth function $\theta : [a,b] \to \mathbb{R}$,

$$\int_a^b \left(r\theta_r^2 + \frac{1}{r}\sin^2\theta \right) dr \geq |\cos\theta(a) - \cos\theta(b)|, \tag{2.6}$$

with equality achieved by the stationary profiles $n\pi \pm \Theta(\cdot;q)$ only ($n \in \mathbb{Z}, q > 0$).

2.1. Energy concentration. We assume that for $0 \leq t < T$ the smooth solution $\theta(r,t)$ defines a smooth solution $u = u(x,y,t)$ such that the center of the disk is mapped to the North Pole, i.e. $\theta(0,t) = 0$, for which, as $t \uparrow T$, the energy concentrates. In view of the fact that, away from $r = 0$, (1.3) is a one-dimensional semi-linear uniformly parabolic equation for $\theta(r,t)$, with a bounded smooth nonlinearity, we are greatly helped by the observation that consequently the solution is smooth in every point

$$(r,t) \in [0,1] \times [0,T] \backslash \{(0,T)\}.$$

We thus observe that energy concentration is in fact only possible at the origin, and equivalent to

$$E(1;\theta(\cdot,T)) + \int_0^T \int_0^1 \theta_t^2 \, r dr dt < E(1;\theta_0), \tag{2.7}$$

meaning

$$\bar\epsilon = E(1;\theta(\cdot,T^-)) - E(1;\theta(\cdot,T)) > 0.$$

In view of the smoothness for $r > 0$ it follows that for all $\epsilon < \bar\epsilon$ and all $0 \leq t < T$ the minimal value $r = R_\epsilon(t)$ for which

$$E(R_\epsilon(t);\theta(\cdot,t)) = \epsilon, \tag{2.8}$$

is well defined and must satisfy, since $\epsilon < \bar\epsilon$,

$$\lim_{t \uparrow T} R_\epsilon(t) = 0. \tag{2.9}$$

For now we assume that $E_r > 0$ at $r = R_\epsilon(t)$ so that $r = R_\epsilon(t)$ is a smooth level curve of $E(r,t)$. In fact E_r is nonnegative and can only be zero if $\sin\theta = \theta_r = 0$, a situation which we will exclude as a byproduct of our reasoning for the ϵ's under consideration. Moreover, the intersection-comparison arguments in Section 3 imply an eventual lower bound of the form $r\theta_r \geq \sin\theta$ to the left of the first intersection of θ with π.

2.2. Parabolic scalings. Following Struwe we introduce, for given t_k and $R_k = R_\epsilon(t_k)$, i.e. with

$$E(R_k;\theta(\cdot,t_k)) = \epsilon \quad (0 < \epsilon < 2), \tag{2.10}$$

scalings v_k of θ defined by

$$v_k(r,t) = \theta(rR_k, t_k + tR_k^2). \tag{2.11}$$

In the reasoning below we shall not use that $r = R_\epsilon(t)$ is a smooth level curve, nor that $E_r > 0$ at $r = R_\epsilon(t_k)$. If we just pick $r = R_\epsilon(t)$ to be a radius r for which $E(r,t) = \epsilon$, there is no real need to worry about $E_r = 0$. At the end we shall exclude $E_r(R_\epsilon(t), t) = 0$ for t close to T, so that eventually $r = R_\epsilon(t)$ is indeed a smooth level curve, with $R_\epsilon(t)$ uniquely defined by $E(R_\epsilon(t), t) = \epsilon$.

The scalings v_k also solve (1.3), rewritten as

$$v_t = v_{rr} + \frac{1}{r} v_r - \frac{\sin 2v}{2r^2}. \tag{2.12}$$

Note the correspondence between

$$Q = \{(r,t) :\, 0 < r \le 1, -1 \le t \le 0\}$$

and

$$Q_k = \{(r,t) :\, 0 < r \le R_k, t_k - R_k^2 \le t \le t_k\}. \tag{2.13}$$

as domains for v_k and θ, whence

$$E(1; v_k(\cdot, 0)) = E(R_k; \theta(\cdot, t_k)) = \epsilon.$$

In other words, at $t = 0$ all v_k have energy ϵ on the r-interval $(0, 1]$. We write $(0, 1]$ instead of $[0, 1]$, to emphasize that energy may disappear in $r = 0$, the main point being to exclude this from happening for the v_k as $k \to \infty$.

2.3. Convergence to equilibria away from the origin. Since the original solution θ is uniformly bounded, the scaled solutions v_k are uniformly bounded on Q. In fact, they are defined and uniformly bounded on every $Q_R^S = \{(r,t) :\, 0 < r \le R, -S \le t \le 0\}$ for k sufficiently large, depending on S and R. By standard Schauder estimates and compactness arguments, v_k is then bounded in the usual Hölder spaces $C^{2+\alpha;1+\frac{\alpha}{2}}(Q^S_{\rho,R})$ on

$$Q^S_{\rho,R} = \{(r,t) :\, \rho \le r \le R, -S \le t \le 0\}$$

with arbitrary $0 < \rho < R$ and $S > 0$, and converges, along suitable subsequences of arbitrary subsequences, to a smooth limit solution $\bar{v}$. This limit solution is easily seen to be defined for all $r > 0$ and all $t \le 0$. We also have, say for $Q = Q^1_{0,1}$ and the original Q_k in (2.13), that

$$\iint_Q v_{kt}^2 \, r dr dt = \iint_{Q_k} \theta_t^2 \, r dr dt \to 0 \tag{2.14}$$

as $k \to \infty$ whence $\bar{v}_t = 0$ on Q and, by obvious reasoning, also for all $r > 0$ and all $t \le 0$. Thus $\bar{v}$ is a stationary solution.

The limit $\bar{v}$ may depend on the subsequence. It is our purpose to show that it is in fact uniquely determined, that it has energy ϵ (not smaller) on the r-interval $(0, 1]$ and that $\bar{v}(r) \to 0$ as $r \downarrow 0$. As a consequence we then have

$$\bar{v}(r) = \pm 2 \arctan qr, \quad \frac{2q^2}{1+q^2} = \epsilon. \tag{2.15}$$

Both signs are possible (though not simultaneously for different convergent subsequences). We proceed to show that, on any domain $Q^S_{\rho,R}$, the sequence v_k converges to $\bar{v}$ as $k \to \infty$.

It will be convenient to set

$$E_k(t) = E(1; v_k(\cdot,t)) = \frac{1}{2}\int_0^1 \left(r v_{kr}(r,t)^2 + \frac{1}{r}\sin^2 v_k(r,t)\right) dr, \tag{2.16}$$

so that, writing (2.2) in differentiated form for v_k,

$$E_k(0) = \epsilon, \quad E_k'(t) = -\int_0^1 v_{kt}^2\, r dr + v_{kt}(1,t) v_{kr}(1,t).$$

In view of (2.14) and the strong convergence to an equilibrium away from $r = 0$, this implies that

$$E_k(t) \to \epsilon \tag{2.17}$$

as $k \to \infty$, along the convergent subsequences under consideration, uniformly in $t \in [-1, 0]$. We wish to show that, as a consequence of the way θ has been scaled to v_k, energy can no longer concentrate at $r = 0$, and that the limiting value $v_k(0,t) = 0$ is preserved in the limit $k \to \infty$. Since we already know that $\bar{v}$ is time-independent, it will be sufficient to do this for $w_k = v_k(\cdot, t_k')$ where t_k' is a suitable sequence in the t-interval $[-1, 0]$.

2.4. Control near the origin. The behaviour near $r = 0$ can be controlled using the *Hamiltonian* H_k defined by

$$2H_k = r^2 v_{kr}^2 - \sin^2 v_k = v_{ks}^2 - \sin^2 v_k, \tag{2.18}$$

which satisfies

$$H_{kr} = r^2 v_{kt} v_{kr}. \tag{2.19}$$

Integrating (2.19) we find

$$|H_k| \le \int_0^r |v_{kt} v_{kr}| r^2 dr \le r \sqrt{\int_0^1 r v_{kr}^2\, dr}\sqrt{\int_0^1 r v_{kt}^2\, dr}. \tag{2.20}$$

We see from (2.4) and (2.11) that in the right hand side of (2.20) the first square root is uniformly bounded (in k and $t \in [-1, 0]$), while the square root of the integral involving the t-derivative is square integrable over $t \in [-1, 0]$, with the integral converging to zero as $k \to \infty$.

2.5. Reduction to spatial variables only. Though we cannot make the right hand side small uniformly in $t \in [-1, 0]$, we can make it uniformly small on subsets with measure close to one. In particular there exist $-1 \le t_k' \le 0$ such that

$$w_k = v_k(\cdot, t_k')$$

satisfies $w_{ks}^2 - \sin^2 w_k = o(\exp(s))$ as $k \to \infty$ (we remind that $s = \log r$). We picture this in the (w, w')-phase plane for the equation

$$w'' = \frac{\sin 2w}{2}.$$

With w and w' also considered as coordinates from here on, the ensemble of stationary solutions connecting neighbouring π-multiples is given by

$$N = \{(w, w') : \, w'^2 = \sin^2 w\}.$$

In particular the connecting orbit $\{(w, w') : \, w' = \sin w, \, 0 < w < \pi\}$ contains the solutions (1.8), i.e. all the shifts of $2 \arctan \exp(s)$.

Clearly (w_k, w_{ks}) is close to $(w, w') = (0, 0)$ for $s \to -\infty$. The key observation now is that, once (w_k, w_{ks}) is away from $(w, w') = (0, 0)$, it must stay close to one of the graphs $w' = \pm \sin w$, which correspond to the stationary solutions connecting the poles, see (1.5). This forces (w_k, w_{ks}) to make a large excursion along which the energy contribution is well tractible, until it comes in the vincinity of either $(w, w') = (\pi, 0)$ or $(w, w') = (-\pi, 0)$, which cannot happen for $s \leq 0$ in view of the energy constraint provided by (2.17) and (2.6).

2.6. No relevant energy contribution near the poles. To force (w_k, w_{ks}) to actually move away from $(w, w') = (0, 0)$, we use an estimate which excludes significant energy contributions near 0 and multiples of π, formulated in terms of

$$e_k^{(a)}(t) = \frac{1}{2} \int_{|\sin v_k(r,t)| \leq |\sin a|} \left(r v_{kr}(r, t)^2 + \frac{1}{r} \sin^2 v_k(r, t) \right) dr. \tag{2.21}$$

With (2.21) the energy (2.16) splits up as

$$E_k(t) = e_k^{(a)}(t) + E_k^{(a)}(t). \tag{2.22}$$

To estimate $e_k^{(a)}(t)$ we test (2.12) with $f_a(v)$, where f_a is the unique continuous piecewise linear odd π-periodic function with

$$f_a(v) = v \quad (|v| \leq a); \quad f_a'(v) = -\frac{2a}{\pi - 2a} \quad (a < v < \pi - a). \tag{2.23}$$

Integrating by parts we obtain for v_k, setting

$$B = B_{k,a,t} = \{0 < r \leq 1 : \, |\sin v_k(r, t)| \leq |\sin a|\}, \quad G = (0, 1] \backslash B$$

and omitting subscripts k, that

$$\int_B v_r^2 \, r dr + \int_0^1 \frac{f_a(v) \sin 2v}{2r} \, r dr = \frac{2a}{\pi - 2a} \int_G v_r^2 \, r dr + [f_a(v) v_r \, r]_0^1 - \int_0^1 f_a(v) v_t \, r dr. \tag{2.24}$$

The first and second term in the right hand side are $O(a)$, uniformly in k and t, while the third term is estimated as

$$\left| \int_0^1 f_a(v) v_t \, r dr \right| \leq \sqrt{\int_0^1 |f_a(v)|^2 \, r dr} \sqrt{\int_0^1 v_t^2 \, r dr},$$

and therefore $O(a)$ in $L^2(-1, 0)$. Note that $f_a(v) \sin 2v \sim 2 \sin^2 v$ for small $\sin v$. We therefore have that, with the same t_k' as above,

$$e_k^{(a)} = e_k^{(a)}(t_k') = \frac{1}{2} \int_{|\sin v_k(r, t_k')| \leq |\sin a|} \left(r v_{kr}(r, t_k')^2 + \frac{1}{r} \sin^2 v_k(r, t_k') \right) dr = O(a),$$

uniformly in k. The fact that we may use the same t_k' is due to the uniform boundedness of $\int_0^1 v_{kr}^2 \, r dr$ and the smallness of $\int_{-1}^1 \int_0^1 v_{kt}^2 \, r dr \, dt$ as $k \to \infty$.

2.7. Convergence with conservation of energy. In view of the above reasoning we continue the argument with w_k, its energy

$$E_k = \frac{1}{2}\int_{-\infty}^{0}(w_{ks}^2 + \sin^2 w_k)ds \tag{2.25}$$

satisfying

$$E_k \to \epsilon, \quad E_k = E_k^{(a)} + e_k^{(a)}, \tag{2.26}$$

with

$$e_k^{(a)} = \frac{1}{2}\int_{|\sin w_k|\leq|\sin a|}(w_{ks}^2 + \sin^2 w_k)ds = O(a) \tag{2.27}$$

uniformly in k, and, since

$$w_{ks}^2 - \sin^2 w_k = o(\exp(s)) \tag{2.28}$$

as $k \to \infty$, it follows, in addition to $(w_k, w_{ks}) \to (0,0)$ as $s \to -\infty$, that for every $\delta > 0$ we have

$$(w_k, w_{ks}) \in N_\delta = \{(w,w') : |w'^2 - \sin^2 w| < \delta\}$$

for k sufficiently large.

Choosing $0 < a << \epsilon$, and then δ sufficiently small, we have, with

$$N_{\delta,a} = \{(w,w') : |w'^2 - \sin^2 w| < \delta, |\sin w| < \sin a\} = \cup_{n=-\infty}^{n=\infty} N_{\delta,a}^n,$$

where

$$N_{\delta,a}^n = \{(w,w') : |w'^2 - \sin^2 w| < \delta, |w - n\pi| < a\},$$

that the boundary $\partial N_{\delta,a}^0$ of $N_{\delta,a}^0$ consists of 4 vertical parts and 4 curved parts. The vertical parts are identified by the signs of $w = \pm a$ and w'.

For k sufficiently large, the curve traced out by (w_k, w_{ks}) lies in $N_{\delta,a}^0$ for s in a neighbourhood of $-\infty$ and, in view of (2.26), must leave $N_{\delta,a}^0$, the point of exit lying either on the right upper vertical part ($w = a$, $w' > 0$) or on the left lower vertical part ($w = -a$, $w' < 0$) of $\partial N_{\delta,a}^0$.

Suppose we are in the first case. Then, for some k-dependent s-value $\underline{s}_0(k) < 0$, we have that $w_k = a$, $w_{ks} > 0$, and (w_k, w_{ks}) is trapped in

$$M_{\delta,a}^{1,+} = \{(w,w') : |w'^2 - \sin^2 w| < \delta, a < w < \pi - a, w' > 0\},$$

until it leaves this set through a point with $w = \pi - a$ and $w' > 0$. It is forced to do so for some finite $s = \bar{s}_0(k)$, because while in $M_{\delta,a}^{1,+}$, the derivative $w_k'(s)$ is positive and bounded away from zero. Note that $\bar{s}_0(k) > 0$, because $\bar{s}_0(k) \leq 0$ is impossible, the energy E_k being too small to allow w_k to go from a to $\pi - a$, see (2.6). Thus $s = \bar{s}_0(k)$ corresponds to some $r > 1$. Since we can choose δ as small as we like, by taking k large (depending on δ), it follows that $w_{ks} - \sin w_k$ can be made uniformly small on the s-interval $[\underline{s}_0(k), \bar{s}_0(k)]$, so that w_k, as a function of s, becomes C^1-close to a possibly k-dependent shift $w(s) = 2\arctan\exp(s + S_k)$ of $2\arctan\exp(s)$. Since we control E_k, this shift is controlled and we claim that w_k becomes C^1-close to $\bar{w}$ defined by $\bar{w}(s) = 2\arctan\exp(s + s_\epsilon)$ where s_ϵ is given by, see (2.5),

$$\epsilon = 1 - \cos 2\arctan\exp(s_\epsilon) = \frac{2\exp(2s_\epsilon)}{1+\exp(2s_\epsilon)}.$$

Equivalently, $S_k \to s_\epsilon$. A larger limit for S_k is impossible in view of (2.6) and a smaller limit is excluded by (2.26), since we may take a arbitrarily small. We conclude that, in terms of r,

$$w_k = w_k(r) \to \bar{w}(r) = \Theta(r, q_\epsilon) = 2\arctan q_\epsilon r, \quad q_\epsilon = \sqrt{\frac{\epsilon}{2-\epsilon}} \tag{2.29}$$

The second case is similar. If (w_k, w_{ks}) leaves $N^0_{\delta,a}$ through a point of exit lying in the left lower part ($w = -a$, $w' < 0$), then, for some $s = \underline{s}_0(k) < 0$, we have that $w_k = -a$, $w_{ks} < 0$, and (w_k, w_{ks}) is trapped in

$$M^{1,-}_{\delta,a} = \{(w, w') : |w'^2 - \sin^2 w| < \delta, -\pi + a < w < -a, w' < 0\},$$

until it leaves this set through $w = -\pi + a$ with $w' < 0$. By identical reasoning, it then follows that

$$w_k = w_k(r) \to \bar{w}(r) = -\Theta(r, q_\epsilon) = -2\arctan q_\epsilon r. \tag{2.30}$$

2.8. Convergence of the scaled profiles. Returning to the sequence v_k defined by (2.11) we observe that with (2.29) and (2.30) we have identified the only two possible limits, in particular also for all possible convergent subsequences of arbitrary sequences of real numbers

$$\theta(R_\epsilon(t_k), t_k) = v_k(1, 0), \tag{2.31}$$

so that we may conclude that for any sequence $t_k \uparrow T$ there is a further subsequence such that

$$\theta(R_\epsilon(t_k), t_k) \to \pm 2\arctan q_\epsilon.$$

Since $\theta(R_\epsilon(t), t)$ is a bounded function of t, this implies that

$$\lim_{t\uparrow T} \theta(R_\epsilon(t), t) = \pm 2\arctan q_\epsilon, \tag{2.32}$$

meaning that the limit in (2.32) exists, and that either of the two values is possible. We now conclude that eventually

$$E_r(R_\epsilon(t), t) > 0, \tag{2.33}$$

so that $R_\epsilon(t)$ is smooth. Finally, reapplying the Schauder/compactness arguments above with the knowledge in (2.32) and (2.33) we conclude that

$$\theta(rR_\epsilon(t), t) \to \pm 2\arctan q_\epsilon r \tag{2.34}$$

as $t \uparrow T$ in $C^m([\rho, R])$ for every $m \in \mathbb{N}$ and every $0 < \rho < R < \infty$.

2.9. Improved convergence and equivalence of scales. In this subsection we show that the convergence in the inner scale, see (2.34) above, is in fact in $C^k([0,R])$ for every $R>0$, i.e., uniform near $r=0$. This immediately implies that

$$\lim_{t\uparrow T}\theta_r(0,t)=\pm\infty,$$

(the divergent limit exists as either $+\infty$ or $-\infty$). Energy concentration as $t\to T$ thus implies the unboundedness of $\theta_r(0,t)=p(0,t)$, where $p(r,t)$ is the radially symmetric solution of the semilinear parabolic equation (1.9) in $\mathbb{R}^4$, in the which the singularity in $r=0$ has been removed.

Let us continue under the assumption that (2.34) holds with a plus sign. So far we have scaled the spatial variable using the energy level curves, see (2.8), which differ from the level curves of the polar angle θ. For the equilibria they coincide in view of the relation between energy and polar angle given by (2.5). As a consequence of (2.34) they are asymptotic to one another as $t\uparrow T$. To be precise, we will establish the uniform convergence near $r=0$ noting that, say with $\epsilon=\frac{1}{2}$, we may use (2.34) to conclude that for $t<T$ close to T, the smooth level curve $(R(t),t)$ is well defined by

$$\theta(R(t),t)=\frac{\pi}{3},$$

and satisfies

$$\lim_{t\uparrow T}\frac{R(t)}{R_{\frac{1}{2}}(t)}=1,\quad \text{i.e. } R(t)\sim R_{\frac{1}{2}}(t).$$

Switching from $R_{\frac{1}{2}}(t)$ to $R(t)$, and scaling accordingly,

$$v(r,t)=\theta(rR(t),t)\to 2\arctan\frac{r}{\sqrt{3}} \tag{2.35}$$

as $t\uparrow T$ in $C^m([\rho,R])$ for every $m\in\mathbb{N}$ and every $0<\rho<R<\infty$. Note that this implies that the energy of $\theta(\cdot,t)$ between $r=R(t)$ and $r=R_{\frac{1}{2}}(t)$ goes to zero, i.e.

$$E(R(t);\theta(\cdot,t))\to\frac{1}{2}.$$

We conclude that

$$RR'=-\left.R\frac{\theta_t}{\theta_r}\right|_{r=R}=-\left.\frac{R^2\theta_{rr}+R\theta_r-\frac{1}{2}\sin 2\theta}{R\theta_r}\right|_{r=R}=-\left.\frac{v_{rr}+v_r-\frac{1}{2}\sin 2v}{v_r}\right|_{r=1}\to 0$$

as $t\uparrow T$. This can now be used in the equation for v, which reads, see also [17],

$$R(t)^2v_t=v_{rr}+\frac{1}{r}v_r-\frac{\sin 2v}{2r^2}+R(t)R'(t)rv_r. \tag{2.36}$$

Absorbing R^2 in a new time variable τ and defining $f=f(\tau)$ by

$$\frac{d\tau}{dt}=\frac{1}{R(t)^2},\quad \dot{R}=\frac{dR}{d\tau},\quad f=\frac{\dot{R}}{R}=RR',$$

we have $\tau\uparrow\infty$ as $t\uparrow T$, simply because $0<R\to 0$ and $(R^2)'\to 0$, and arrive at

$$v_\tau=v_{rr}+\frac{1}{r}v_r-\frac{\sin 2v}{2r^2}+f(\tau)rv_r. \tag{2.37}$$

The smooth coefficient $f(\tau)$ satisfies $f(\tau) \to 0$ as $\tau \uparrow \infty$. As a function of r and τ the scaled solution $v(r,\tau)$ satisfies of course the boundary condition $v(1,\tau) = \frac{\pi}{3}$, but in addition we know that, as $\tau \uparrow \infty$,

$$v(r,\tau) \to 2 \arctan \frac{r}{\sqrt{3}}.$$

The convergence holds in every $C^m([\rho,1])$ with $m \in \mathbb{N}$ and $0 < \rho < 1$. Extending this result to convergence in $C^m([0,1])$ will immediately imply that asymptotically all three scales are the same:

$$R_{BHK}(t) \sim R(t) \sim R_{\frac{1}{2}}(t).$$

To extend this result to $C^m([0,1])$, we remove the singularity by setting

$$v(r,\tau) = rp(r,\tau),$$

so that, see also (1.9), the equation for p becomes

$$p_\tau = p_{rr} + \frac{3}{r} p_r + F(r,p) + f(\tau)(rp_r + p), \quad F(r,p) = \frac{1}{r^2}(p - \frac{\sin 2rp}{2r}), \tag{2.38}$$

with boundary condition

$$p(1,\tau) = \frac{\pi}{3}. \tag{2.39}$$

As in (1.9) the nonlinear term $F(r,p)$ in (2.38) is smooth since it expands as

$$F(r,p) = \frac{2}{3} p^3 - \frac{2}{15} r^2 p^5 + \cdots .$$

By standard arguments (2.38,2.39) has a unique global classical smooth solution converging in any $C^m([0,1])$ as $\tau \uparrow \infty$, provided $p(r,\tau)$ is controled by a uniform a priori bound. It remains to establish such a bound.

We already know that $v(r,\tau)$ is smooth with $v(0,\tau) = 0$, and it is clear that $|v(r,\tau)|$ is bounded by a constant which is asymptotically equal to $\frac{\pi}{3}$, in view of the obvious energy considerations, which also control $v(r,\tau)$ for small r. Certainly we may then conclude that

$$|v(r,\tau)| < \frac{\pi}{2}, \tag{2.40}$$

which, in the original variables, means that

$$|\theta(r,t)| < \frac{\pi}{2} \quad \text{for} \ \ 0 \le r \le R(t), \tag{2.41}$$

for all t sufficiently close to T. We choose such a t_0 and then q with $2 \arctan q > \frac{\pi}{2}$ such that

$$-2 \arctan \left(\frac{qr}{R(t_0)} \right) \le \theta(r,t_0) \le 2 \arctan \left(\frac{qr}{R(t_0)} \right) \quad \text{on} \ \ [0, R(t_0)]. \tag{2.42}$$

Consider any $t_1 \in (t_0, T)$ with $R(t_1) \le R(t_0)$. Then (2.42) may be replaced by

$$-2 \arctan \left(\frac{qr}{R(t_1)} \right) \le \theta(r,t_0) \le 2 \arctan \left(\frac{qr}{R(t_1)} \right) \quad \text{on} \ \ [0, R(t_0)], \tag{2.43}$$

and clearly we have, for $t_0 \le t \le t_1$ that

$$-2\arctan\left(\frac{qr}{R(t_1)}\right) \le \theta(r,t) \le 2\arctan\left(\frac{qr}{R(t_1)}\right) \quad \text{in} \quad r = R(t_0). \tag{2.44}$$

Applying the comparison principle for smooth solutions of the original θ-equation (1.3) on the rectangle $[0, R(t_0)] \times [t_0, t_1]$, we conclude from (2.43,2.44) that

$$-2\arctan\left(\frac{qr}{R(t_1)}\right) \le \theta(r,t_1) \le 2\arctan\left(\frac{qr}{R(t_1)}\right) \quad \text{for} \quad r \in [0, R(t_0)]. \tag{2.45}$$

In terms of $v(r,\tau)$, with τ_0 and τ_1 corresponding to t_0 and t_1, we conclude from (2.45) that

$$-2\arctan qr \le v(r,\tau_1) \le 2\arctan qr \quad \text{for} \quad r \in [0,1]. \tag{2.46}$$

We only used that $R(\tau_1) \le R(\tau_0)$ to prove (2.46). Since $R(\tau) \to 0$ the desired a priori bound has been established.

2.10. Equivalent descriptions of blow-up. In the previous subsection we obtained, as a corollary of our improved convergence result in the inner energy scale, that $p(0,t) = \theta_r(0,t)$ diverges to $\pm\infty$ if the energy concentrates. In particular this implies that $p(0,t) = \theta_r(0,t)$ is unbounded as $t \uparrow T$. It is not clear to us whether the blow-up of θ_r was actually claimed or proved in [8], since such properties of a subsolution do not necessarily carry over to a solution, in view of the blow-up times being different.

For the opposite implication (blow-up of θ_r implies energy concentration) we observe that an unbounded θ_r forces a first intersection with $\pm\pi$, say $S(t)$ defined by $\theta(S(t),t) = \pm\pi$ to satisfy

$$\liminf_{t \uparrow T} S(t) = 0. \tag{2.47}$$

Otherwise $|\theta(r,t)| \le \pi$ in some rectangle $[0, r_0] \times [t_0, T)$ and this allows a construction of global super- en subsolutions $\bar\theta$ and $\underline{\theta}$ and with $\bar\theta(0,t) = \pi$ and $\underline{\theta}(0,t) = -\pi$, and a subsequent barrier argument preventing blow-up of θ_r, see [7]. We conclude that blow-up of θ_r implies (2.47) and hence energy concentration. Energy concentration in turn implies that $S(t) \to 0$ as $t \uparrow T$ along at least one subsequence, so all three characterisations of singularity formulation are, in the end, equivalent. Using the real analyticity of $\theta(r,T)$ in $r > 0$, which may be established using arguments presented to us by Angenent, one concludes that the lim inf in (2.47) may be replaced by a limit symbol. Otherwise $\theta(\cdot,T)$ would be identically equal to π.

2.11. Scalings for larger energy drops. Clearly (2.34) implies that the energy jump is at least equal to 2, the energy quantum defined by the function $2\arctan$ in (1.8). Suppose it is larger. Denoting the previously used ϵ by ϵ_0, we proceed with, see (2.9, 2.10, 2.11), some $R_{2+\epsilon_1}(t)$ for which

$$E(R_{2+\epsilon_1}(t); \theta(\cdot,t)) = 2 + \epsilon_1,$$

with $0 < \epsilon_1 < 2$, and repeat the argument above. In fact we may carry out the argument simultaneously for all j for which $R_{2j+\epsilon_j}(t)$, with $0 < \epsilon_j < 2$ may be chosen such that

$$E(R_{2j+\epsilon_j}(t); \theta(\cdot,t)) = 2j + \epsilon_j$$

and satisfies $R_{2j+\epsilon_j}(t) \to 0$ as $t \uparrow T$. Clearly j runs from $j=0$ (the case treated above) to some finite maximal $j=J$. Thus for any given sequence $t_k \uparrow T$ and corresponding $R_k^{(j)} = R_{2j+\epsilon_j}(t_k)$, we define $v_k^{(j)}$ by

$$v_k^{(j)}(r,t) = \theta(rR_k^{(j)}, t_k + t(R_k^{(j)})^2).$$

Provided $R_{2(j-1)\epsilon_{j-1}}(t) << R_{2(j)+\epsilon_j}(t)$, the scalings

$$v_k^{(0)}, \dots v_k^{(J)}$$

will have nothing in common, except at $t=0$, where $v_k^{(j)}(r,0) = \theta(rR_k^{(j)}, t_k)$ will contain $v_k^{(j-1)}(r,0) = \theta(rR_k^{(j-1)}, t_k)$ as an inner layer near $r=0$. This determines the line of reasoning below.

2.12. Compactness and reduction to spatial variables. We fix $0 \le j \le J$. At $t=0$, the scalings $v_k^{(j)}$ have energy $2j+\epsilon_j$ on the r-interval $(0,1]$. Again the sequence $v_k^{(j)}$ is bounded in the usual Hölder spaces and converges, along a suitable subsequences of arbitrary subsequences, to a smooth stationary limit solution $\bar{v}^{(j)}$ defined for all $r>0$ and all $t \le 0$, a limit which, a priori, may again depend on the subsequence.

As before, the energy satisfies

$$E_k^{(j)}(t) = E(1; v_k^{(j)}(\cdot,t)) \to 2j + \epsilon_j$$

as $k \to \infty$, along the same subsequences, uniformly in $t \in [-1,0]$. Again we can choose $-1 \le t_k' \le 0$ such that $w_k^{(j)} = v_k^{(j)}(\cdot, t_k')$ has $(w_k^{(j)}, w_{ks}^{(j)}) \in N_\delta$ for k sufficiently large, with $(w_k^{(j)}, w_{ks}^{(j)})$ close to $(w,w') = (0,0)$ for $s \to -\infty$, and the energy

$$E_k^{(j)} = E_k^{(j)}(t_k') = \frac{1}{2}\int_{-\infty}^{0} ((w_{ks}^{(j)})^2 + \sin^2 w_k^{(j)})ds$$

satisfying

$$E_k^{(j)} \to \epsilon, \quad E_k^{(j)} = E_k^{(ja)} + e_k^{(ja)},$$

with

$$e_k^{(ja)} = \frac{1}{2}\int_{|\sin w_k| \le |\sin a|} ((w_{ks}^{(j)})^2 + \sin^2 w_k^{(j)})ds = O(a)$$

uniformly in k, and, as $k \to \infty$,

$$w_{ks}^2 - \sin^2 w_k = o(\exp(s)).$$

2.13. Convergence modulo logarithmic shifts. We recall that

$$N_{\delta,a}^n = \{(w,w') : |w'^2 - \sin^2 w| < \delta, \, |w - n\pi| < a\},$$

and introduce

$$M_{\delta,a}^{n,+} = \{(w,w') : |w'^2 - \sin^2 w| < \delta, \, (n-1)\pi + a < w < n\pi - a, \, w' > 0\},$$

$$M^{n,-}_{\delta,a} = \{(w, w') : |w'^2 - \sin^2 w| < \delta, -n\pi + a < w < -(n-1)\pi - a, w' < 0\},$$

trapping regions for each connecting orbits in the (w, w')-phase plane.

With (all) $0 < \epsilon_j < 2$ already given, we may choose $a > 0$ as small as we like, and having chosen a, we may subsequently choose $\delta > 0$ as small as we like, as we did above, so that in particular every boundary $\partial N^j_{\delta,a}$ consists of 4 vertical parts, each of which is identified by a choice of the signs in $w - n\pi = \pm a$ and w', and 4 curved parts. Since $(w^{(j)}_k, w^{(j)}_{ks})$ is close to $(w, w') = (0, 0)$ for $s \to -\infty$, it starts off in $N^0_{\delta,a}$. Copying the previous argument for $(w^{(0)}_k, w^{(0)}_{ks})$, it must either exit $N^0_{\delta,a}$ entering $M^{1,+}_{\delta,a}$ at some $s = \underline{s}^{(j)}_0(k) < 0$, to later exit $M^{1,+}_{\delta,a}$ and enter $N^1_{\delta,a}$ at some $s = \bar{s}^{(j)}_0(k)$, or it must exit $N^0_{\delta,a}$ entering $M^{1,-}_{\delta,a}$ at some $s = \underline{s}^{(j)}_0(k) < 0$, to later exit $M^{1,-}_{\delta,a}$ and enter $N^{-1}_{\delta,a}$ at some $s = \bar{s}^{(j)}_0(k)$. Unless $j = 0$, we have in either case that $\bar{s}^{(j)}_0(k) < 0$ for k sufficiently large, because otherwise we would have that $\liminf_{k\to\infty} E^{(j)}_k \le 2$, contradicting the definition of $E^{(j)}_k$. For the same reason we have that, in the first case, $(w^{(j)}_k, w^{(j)}_{ks})$ must exit $N^1_{\delta,a}$ entering either $M^{2,+}_{\delta,a}$ or $M^{0,-}_{\delta,a}$ at some $s = \underline{s}^{(j)}_1(k) < 0$, while in the second case, $(w^{(j)}_k, w^{(j)}_{ks})$ must exit $N^{-1}_{\delta,a}$ entering either $M^{0,+}_{\delta,a}$ or $M^{2,-}_{\delta,a}$ at some $s = \underline{s}^{(j)}_1(k) < 0$. Continuing the reasoning we obtain

$$\underline{s}^{(j)}_0(k) < \bar{s}^{(j)}_0(k) < \underline{s}^{(j)}_1(k) < \cdots < 0 < \bar{s}^{(j)}_j(k),$$

a sequence of exit and entrance values. Numbering $i = 0, \ldots, j$, each $\underline{s}^{(j)}_i(k)$ is the s-value at which $(w^{(j)}_k, w^{(j)}_{ks})$ exits $N^{n_{i,j}}_{\delta,a}$ and each $\bar{s}^{(j)}_i(k)$ is the s-value at which $(w^{(j)}_k, w^{(j)}_{ks})$ enters $N^{n_{i+1,j}}_{\delta,a}$, with $n_{i+1,j} = n_{i,j} \pm 1$. Only the last s-value $s = \bar{s}^{(j)}_j(k)$ corresponds to some $r > 1$.

Since we can choose δ as small as we like, by taking k large (depending on δ), it follows that, on each s-interval $[\underline{s}_i(k), \bar{s}_i(k)]$, either $w^{(j)}_{ks} - \sin w^{(j)}_k$ or $w^{(j)}_{ks} + \sin w^{(j)}_k$ can be made uniformly small, so that on each of these s-intervals, modulo π, $w^{(j)}_k$ becomes C^1-close to a shift $\pm 2 \arctan \exp(s + S_{i,j,k})$ of $\pm 2 \arctan \exp(s)$.

2.14. Convergence and conservation of energy for the outer layers. The control on $E^{(j)}_k$ forces that $S_{i,j,k} \to -\infty$ as $k \to \infty$ for $i = 0, \ldots, j-1$ and that $S_{j,j,k} \to s_{\epsilon_j}$, with, as before

$$\epsilon_j = 1 - \cos 2 \arctan \exp(s_{\epsilon_j}).$$

Thus in each scale the inner layers dissappear and only one outer layer survives, its limit having precisely energy ϵ_j. Moreover, the shifts really differ in the sense that also $S_{i,j,k} - S_{i-1,j,k} \to \infty$ as $k \to \infty$, because there are no *multiple loop* stationary solutions. Independence of the choice of sequences and subsequences requires no additional arguments than the ones already given for $j = 0$. Thus we conclude that

$$\theta(rR_{2j+\epsilon_j}(t), t) \to m_j\pi + 2(-1)^{n_j} \arctan q_{\epsilon_j} r \tag{2.48}$$

as $t \uparrow T$, uniformly on compact intervals $[a, b] \subset \mathbb{R}^+$. The convergence in (2.48) is easily seen to hold in every $C^k([a, b])$).

2.15. The bubble tree result. Since we already know from (2.34) that $m_0 = 0$, we conclude, a posteriori, with the observation that

$$m_i = m_{i-1} + (-1)^{n_i}, \quad (i = 1, \ldots, j), \tag{2.49}$$

simplify because every violation of (2.49) would require at least another quantum. This completes the proof of the bubble tree result.

3. Eventual monotonicity of the scaling $R_{\rm BHK}(t)$. We first prove that there exist $\alpha_0 > 0$ and $t_1 \in (0, T)$ such that

$$\begin{gathered}\theta(\cdot, t) \text{ and } \Theta(\cdot;\alpha) \text{ have at most one intersection point in } (0, S(t)) \\ \text{for all } t_1 < t < T \text{ and } \alpha > \alpha_0\end{gathered} \tag{3.1}$$

and

$$\theta(r,t) > \Theta(r;\alpha_0) \quad \text{for all } 0 < r \le S(t) \text{ and } t_1 < t < T. \tag{3.2}$$

Indeed, let $0 < \tilde t < T$ be such that $0 < S(\tilde t) = \min_{t\le \tilde t} S(t)$. By the parabolic boundary point lemma, $\theta_r(S(\tilde t),\tilde t) > 0$ and hence there exists $\alpha_0 > 0$ such that $\theta(\cdot,\tilde t)$ and $\Theta(\cdot;\alpha)$ have one intersection point in $(0, S(\tilde t))$ if $\alpha \ge \alpha_0$. A simple intersection-comparison argument shows that the number of intersection points of θ and $\Theta(\cdot;\alpha)$ in $(0, S(\tilde t))$ to the left is nonincreasing in t, and (3.1) holds for any $t_1 \in [\tilde t, T)$. Observe that $\theta_r(0,t) \le \Theta'(0;\alpha_0)$ as long as $\theta(t)$ and $\Theta(\cdot;\alpha_0)$ have one intersection point in $(0, S(t))$. Since $\theta_r(0,t) \to \infty$ as $t \uparrow T$ (see Section 2.10), this implies that (3.2) holds for some $t_1 \in (\tilde t, T)$.

We define, for $t_1 < t < T$,

$$\alpha(t) = \sup\{\alpha \ge \alpha_0 : \theta(r,t) > \Theta(r;\alpha) \quad \text{for all } 0 < r \le S(t)\}.$$

Clearly $\alpha(t) < \infty$, $\theta(\cdot,t) \ge \Theta(\cdot;\alpha(t))$ in $[0, S(t)]$ and

$$\alpha(t) \text{ is nondecreasing in } (t_1, T). \tag{3.3}$$

We claim that

$$\theta_r(0,t) = \Theta'(0;\alpha(t)). \tag{3.4}$$

Arguing by contradiction we suppose that $\theta_r(0,t) > \Theta'(0,\alpha(t))$. Then, by the definition of $\alpha(t)$, there exists $\tilde r \in (0, S(t))$ such that $\theta(\tilde r, t) = \Theta(\tilde r;\alpha(t))$. But then there exists $\tilde\alpha > \alpha(t)$ (with $\tilde\alpha - \alpha(t)$ sufficiently small) such that $\theta(\cdot,t)$ and $\Theta(\cdot,\tilde\alpha)$ have at least 2 intersection points, and we have found a contradiction.

Since $R_{\rm BHK}(t)\theta_r(0,t) = 2$, it follows from (3.3) and (3.4) that

$$R_{\rm BHK}(t) \text{ is decreasing in } (t_1, T).$$

We conclude this section with the result announced in Section 2.1:

$$r\theta_r(r,t) \ge \sin(\theta(r,t)) > 0 \text{ in } (0, S(t)) \text{ if } t_1 < t < T. \tag{3.5}$$

Since $\Theta(r;\alpha)$ satisfies (3.5) with equality, this follows easily from (3.1) and (3.2). Indeed, arguing by contradiction we suppose that $r\theta_r < \sin\theta$ at (r_2, t_2) for some $0 < r_2 < S(t)$ and $t_1 < t < T$. Choosing $\alpha > \alpha_0$ such that (r_2, t_2) is an intersection point of $\theta(\cdot, t_2)$ and $\Theta(\cdot;\alpha)$, there exists $r_3 \in (r_2, S(t_2))$ such that $\theta(r_3, t_2) < \Theta(r_3, t_2)$. Since $\theta(S(t_2), t_2) = \pi > \Theta(S(t_2), t_2)$, there exists a second intersection point of $\theta(\cdot, t_2)$ and $\Theta(\cdot;\alpha)$, and we obtain a contradiction from (3.1).

Observe that (3.5) implies that

$$\theta_r(\cdot,t) > 0 \text{ in } (0, S(t)) \text{ if } t_1 < t < T.$$

4. Only a single jump. In this section we give a simple proof of the result in [19] that in the radially symmetric setting only single jumps can occur. The proof is based on the maximum principle, intersection numbers, barrier functions and a time shift. We use the comparison principle for classical solutions which are continuous down to $r = 0$, see [4].

Let T be the first blow-up time and assume, without loss of generality, that the harmonic bubble which splits off in the *first energy scaling* connects $\theta = 0$, the value of θ at $r = 0$, to $\theta = \pi$. This amounts to having a plus sign in (2.34), so the first lap in the inner scale is from 0 to π. The main result in this section is then:

$$\text{\textit{At most one energy quantum (of amount 2) can concentrate in the origin.}} \tag{4.1}$$

In other words, bubble trees consist of single bubbles. The first lap is the final lap.

To prove this we will use the simple fact that, shifting the solution $\theta(r, t)$ a little bit in time and adding or subtracting π, the following inequalities are immediate from the comparison principle for all τ sufficiently small as soon as $t \geq \tau$:

$$\theta(r, t - \tau) - \pi \leq \theta(r, t) \leq \theta(r, t - \tau) + \pi \quad \text{for all } 0 < r \leq 1 \tag{4.2}$$

We use both inequalities in (4.2), beginning with the second, observing that $\theta(r, t)$ is smooth in $Q = [0, 1] \times [0, T] \setminus (0, T)$, with $\theta(0, t) = 0$ for $0 \leq t < T$. All r and t below are restricted to this set Q. We claim that in view of the blow-up assumption we must have

$$\limsup_{(r,t)\to(0,T)} \theta(r, t) \geq \pi, \tag{4.3}$$

since $\theta(r, t)$ must achieve values larger than or equal to π near $(r, t) = (0, T)$. Otherwise we can use, for sufficiently large values of α, $\Theta(r; \alpha)$ as a supersolution and prevent blow-up with a plus sign in (2.34) as $t \uparrow T$. Now the second inequality in (4.2) prevents a second inner scale lap from π to 2π while strict inequality in (4.3) would force it, in view of the required inner scale energy then being larger than that of a single quantum. Thus

$$\limsup_{(r,t)\to(0,T)} \theta(r, t) = \pi. \tag{4.4}$$

In view of (4.4), in our particular situation a second inner scale energy quantum can only correspond to a second lap from π to 0, i.e. in the second energy scaling the limit profile is $\pi - 2 \arctan r$. Arguing by contradiction we assume that such a second lap exists. We observe that in view of the results in Section 2.9, the first two level curves $R_1(t)$ and $R_2(t)$ for which $\theta(R_1(t), t) = \theta(R_2(t), t) = \frac{\pi}{2}$ are well-defined and smooth for t close to T, and that $\theta(r, t)$ must achieve values larger than π in between. Spatially scaling with $R_1(t)$ and $R_2(t)$, which converge to 0 both, the solution $\theta(\cdot, t)$ converges to, respectively, $\Theta(\cdot; 1)$ and $\pi - \Theta(\cdot; 1)$. We denote the spatially scaled solutions by v_1 and v_2, so $v_i(r, t) = \theta(rR_i(t), t)$. This preparation facilitates the reasoning below with intersections of $\theta(\cdot, t)$ and χ_α defined by

$$\chi_\alpha(r) := \pi - \Theta(r; \alpha).$$

The latter are the equilibria connecting π to 0 in the original unscaled variable r.

The presence of the parameter α allows to use Sard's Lemma and conclude that for almost all $\alpha > 0$ the set

$$\Gamma_\alpha = \{(r, t) |\, 0 < r \leq 1, 0 < t < T, \, \theta(r, t) = \chi_\alpha(r)\}$$

is smooth, meaning that in every point (r_0, t_0) of Γ_α we either have

$$\theta_r(r_0,t_0) - \chi'_\alpha(r_0) \neq 0 \quad \text{or} \quad \theta_r(r_0,t_0) - \chi'_\alpha(r_0) = 0 \neq \theta_t(r_0,t_0) = \theta_{rr}(r_0,t_0) - \chi''_\alpha(r_0)$$

(more precisely, Sard's Lemma is applied to the function $F(r,t) = r \tan \frac{\theta(r,t)}{2}$ and Γ_α is the level set $F(r,t) = \frac{1}{\alpha}$). In the first case we can locally write Γ_α as a smooth curve $r = r_\alpha(t)$, while in the second case we have locally $t = t_\alpha(r)$, with $t'_\alpha(r_0) = 0$ and, in view of the comparison principle, $t''_\alpha(r_0) < 0$ (observe that $t''_\alpha(r_0) \neq 0$ since $\theta_{rr}(r_0,t_0) \neq \chi''_\alpha(r_0)$), so that crossing $t = t_0$ this part of Γ_α disappears.

Avoiding the boundary condition θ_1, by excluding just one value of α for χ_α, we thus have for almost all $\alpha > 0$ that eventually there are a fixed finite number of transversal intersections of $\theta(\cdot, t)$ and χ_α in $(0,1)$. In particular a second intersection $r = z_\alpha(t)$, if it exists, defines a smooth curve defined on some nonempty open interval (t_α, T).

Next we show that the second intersection must exist and pinpoint where it must be situated. Given a t close to T for which $v_1(\cdot,t)$ and $v_2(\cdot,t)$ are already close to $\Theta(\cdot;1)$ and χ_1, we may choose such regular α as above so large that the first intersection occurs for $r < R_1(t)$ at some level below $\frac{\pi}{2}$. This intersection is unique because $\theta(\cdot,t)$ in monotone along its first lap from 0 to π, see Section 3. In particular this requires α to be at least of order $\frac{1}{R_1(t)}$. A second intersection $r = z_\alpha(t)$ can only occur for $r > R_2(t)$, because χ_α is decreasing and we savely can assume that $\theta(r,t) > \frac{\pi}{2}$ for $r \in (R_1(t), R_2(t))$. Here we rely again on the improved convergence in Section 2.9 and Section 2.14. In fact, the second intersection must occur, because otherwise there are no intersections for $r > R_1(t)$ so that in this r-range the solution is bounded below by χ_α, a situation which, in view of the comparison principle, then persists for all further t as long as $t < T$, preventing v_2 from converging to χ_1. Thus we must a have a smooth second intersection curve $r = z_\alpha(t) > R_2(t)$ for almost every α sufficiently large, defined on some interval (t_α, T).

Next we combine this information with the first inequality in (4.2) which we use to find and fix a τ sufficiently small to ensure that

$$\theta(r,t) \geq \theta(r, T-\tau) - \pi = g(r), \quad 0 < r \leq 1, \quad \tau \leq t < T. \tag{4.5}$$

Here $g(r)$ achieves strictly positive values, somewhere between $r = R_1(T-\tau)$ and $r = R_2(T-\tau)$, say in r^*. In particular we may choose α so large that χ_α is below g in $r = r^*$. Hence

$$\theta(r^*,t) > g(r^*) > \chi_\alpha(r^*), \quad \tau \leq t \leq T.$$

Combining with the reasoning above this implies that for almost all α sufficiently large the eventual curve $r = z_\alpha(t)$ lies to the right of r^*. Again this prevents the second θ-lap from π to 0 corresponding to v_2 to concentrate near the origin and this completes the proof that the second inner lap cannot occur.

Remark. It follows at once from (4.1) that

$$\lim_{r\to 0} \theta(r,T) = \pi \quad \text{and} \quad \liminf_{(r,t)\to(0,0)\ (t\leq T)} \theta(r,t) = 0.$$

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