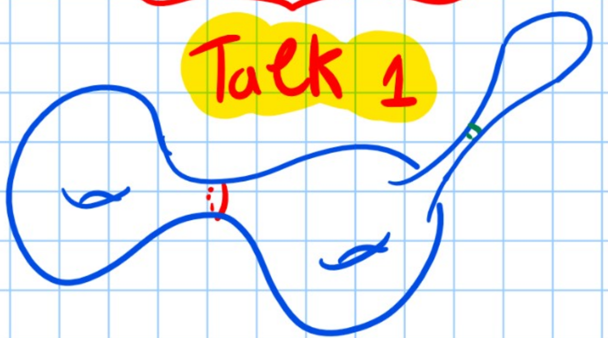
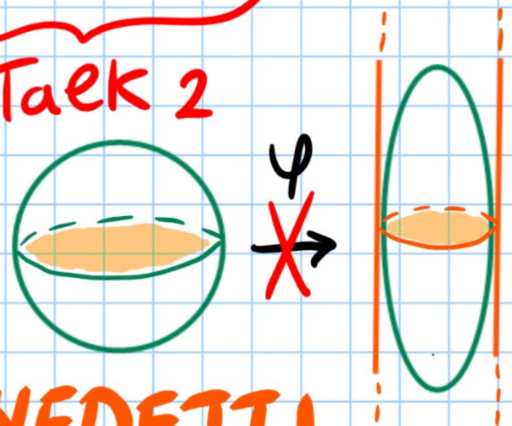
# First Steps into the World of Systolic Inequalities

## From $\underbrace{\text{Riemannian}}_{\text{Talk 1}}$ to $\underbrace{\text{Symplectic}}_{\text{Talk 2}}$ Geometry







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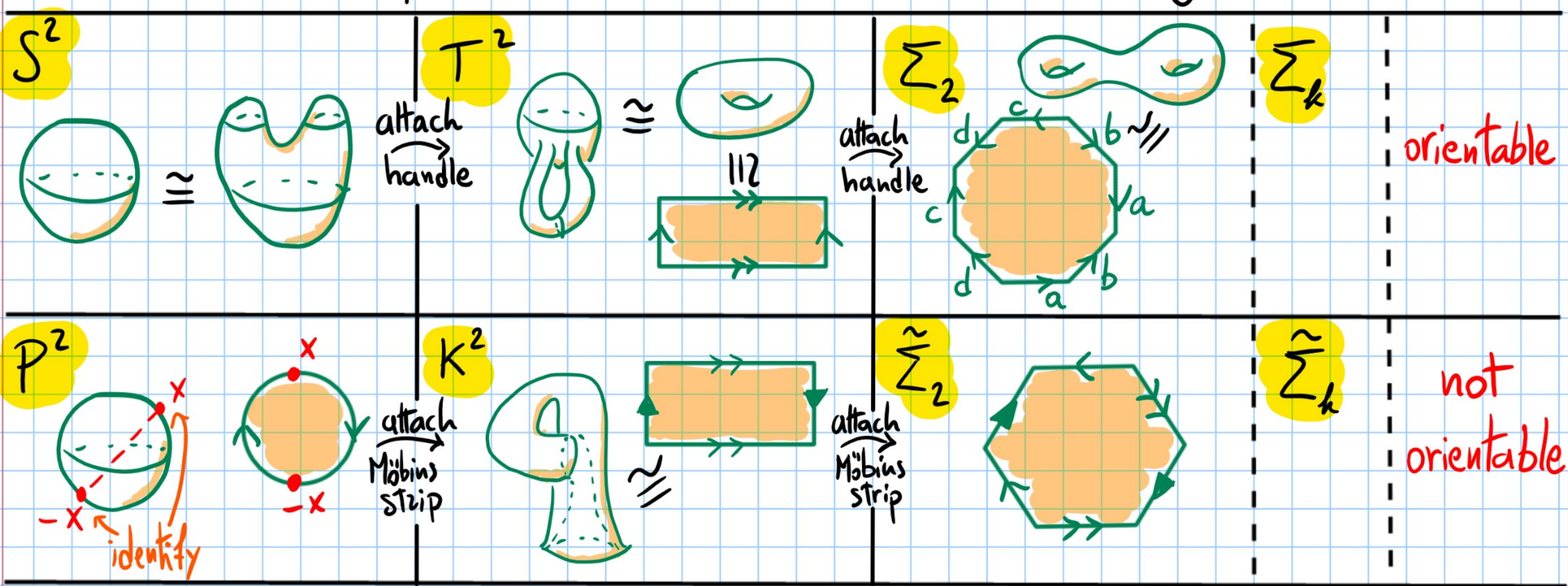
## 1. Riemannian Geometry on Surfaces

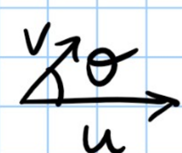
$\Sigma$ surface, compact, without boundary connected. Topological classification:




$g$ Riemannian metric on $\Sigma$ (inner product on each tangent plane).

We can measure things: - lengths $\quad \gamma:[a,b]\to\Sigma, \quad \ell_g(\gamma)=\int_a^b|\dot\gamma|_g dt.$

- angles $\quad \cos\theta=\dfrac{g(u,v)}{|u|_g|v|_g}$

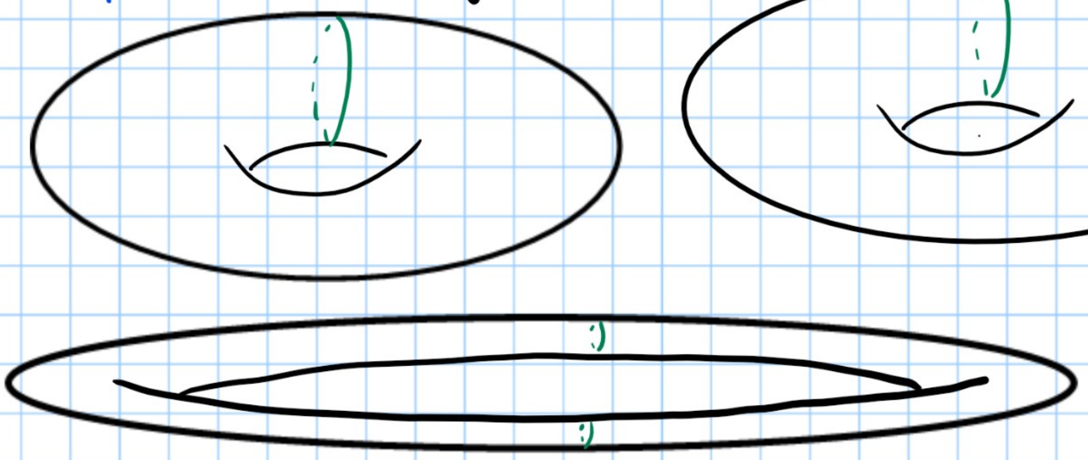


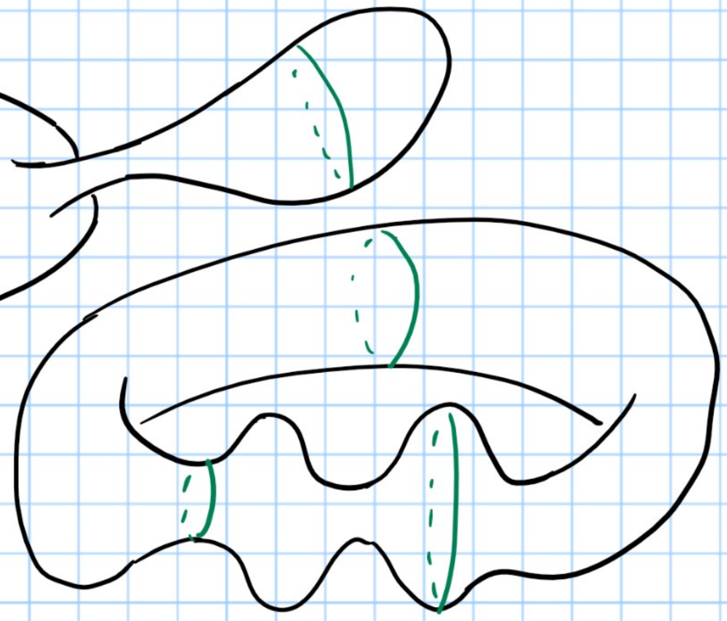

- areas $\quad U\subset\Sigma$ open, $\quad \mathrm{area}_g(U)=\int_U d\mathrm{area}_g.$

$g$ not unique! Define $R(\Sigma):=\{g \text{ Riem. metric on } \Sigma\}/_{\text{scaling}}^{\text{isometries}}$.

$R(\Sigma)$ is huge! Many ways to embed $\Sigma$ in $\mathbb{R}^3$. Restriction of euclidean metric from $\mathbb{R}^3$ to $\Sigma$ yields element of $R(\Sigma)$.

**Example** Embeddings of $T^2$ in $\mathbb{R}^3$





## How to study $\mathcal{R}(\Sigma)$?

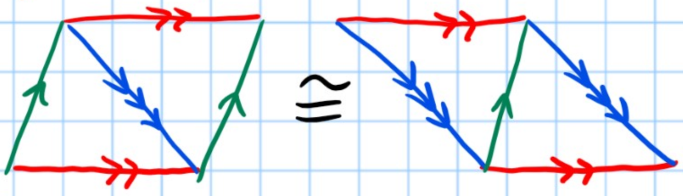
A) Look for special (e.g. symmetric) metrics $g_0 \in \mathcal{R}(\Sigma)$.

B) Compute the size (length or area) w.r.t. $g \in \mathcal{R}(\Sigma)$ of objects in $\Sigma$ (curves or regions).

For A) we consider $g_0$ of constant curvature $K$. Gauß-Bonnet $\Rightarrow$ sign of $K$.

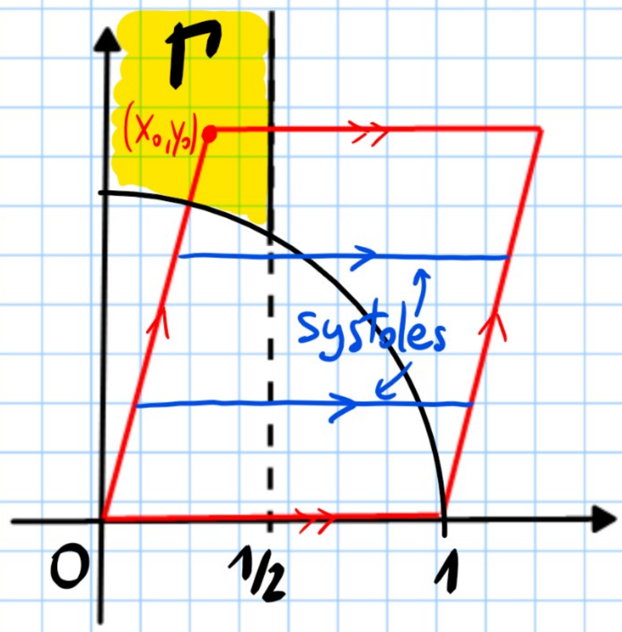
i) On $S^2, P^2$: $K>0$. $g_0$ is unique (round sphere in $\mathbb{R}^3$).

ii) On $T^2, K^2$: $K=0$. We consider $T^2$ only. All possible $g_0$ obtained identifying opposite sides of parallelogram $P$ by translation.



**Problem:** How to classify them? $P$ not unique!

**Thm 1** Every flat torus $g_0$ is obtained from a unique parallelogram with vertices $(0,0), (1,0), (x_0,y_0), (x_0+1, y_0) \in \mathbb{R}^2$, where $(x_0,y_0) \in \Gamma := \{x^2+y^2 \geq 1,\ 0 \leq x \leq 1/2,\ y \geq 0\} \subset \mathbb{R}^2$.




iii) On $\Sigma_k, \tilde{\Sigma}_k$: $K<0$. Cool and deep mathematics to classify possible $g_0$.

**Thm 2** (Uniformization) For every $g \in \mathcal{R}(\Sigma)$ exists $g_0$ of constant curvature and $f: \Sigma \to (0,\infty)$ such that $g = f^2 g_0$. ($g$ conformally equivalent to $g_0$)

For B) we look at...

# 2. Systoles

**Thm 3 (Hadamard)** Let $\Sigma$ be non-simply connected ($\Sigma \neq S^2$). There exists a non-contractible loop $\gamma$ of minimal length (among non-contractible loops). Any such $\gamma$ is a periodic geodesic
($\gamma : \mathbb{R} \to \Sigma$, $\exists\, T > 0 : \gamma(t+T) = \gamma(t)\ \forall t \in \mathbb{R}$, $\ddot{\gamma} = 0$).

acceleration measured using $g$

**Def** Any curve as in Thm 3 is called **systole** of $g$. We denote by **$\mathrm{sys}(g)$** its length.

**Question (qualitative)** Can we have a **long** systole on a $\underbrace{\textbf{small}}_{}$ manifold?

$\mathrm{diam}_g(\Sigma), \mathrm{area}_g(\Sigma)$

**Question (quantitative)**

$$\boxed{\exists\, C > 0 \text{ such that } \sigma(g) := \frac{\mathrm{sys}(g)^2}{\mathrm{area}_g(\Sigma)} \leq C \quad \forall g \in \mathcal{R}(\Sigma) \text{ ?}}$$

If yes, what is the **optimal constant** $C$? Are there **metrics maximizing $\sigma$**?

**Thm 4 (Loewner)** $\sigma(g) \leq \frac{2}{\sqrt{3}}\ \forall g \in T^2$ and $\sigma(g) = \frac{2}{\sqrt{3}}$ iff $g = g_* :=$ .

**Proof** $g = f^2 \cdot g_0$ by Thm 1, where $g_0$ given by Thm 2. Two steps:

i) $\sigma(g) \leq \sigma(g_0)$ and $\sigma(g) = \sigma(g_0)$ iff $f$ is constant.

ii) $\sigma(g_0) \leq \sigma(g_*)$ and $\sigma(g_0) = \sigma(g_*)$ iff $g_0 = g_*$.

$$\ell_g(s \mapsto (s,t)) \geq \mathrm{sys}(g)$$

Let's do i) if $g_0 =$ , $a \geq 1$: $\mathrm{area}_g(T^2) = \int_0^a \left[\int_0^1 f^2(s,t)\,ds\right] dt \geq \int_0^a \left[\int_0^1 f(s,t)\,ds\right]^2 dt$

$$\geq \frac{1 \cdot a}{1^2} \cdot \mathrm{sys}(g)^2 = \mathrm{sys}(g)^2 \cdot \sigma(g_0)^{-1}. \qquad \square$$

**Remark** Thm 4 followed from 1) a normal form for $g$ (uniformization).
2) AM-QM on a family of systoles for $g_0$ foliating $T^2$.

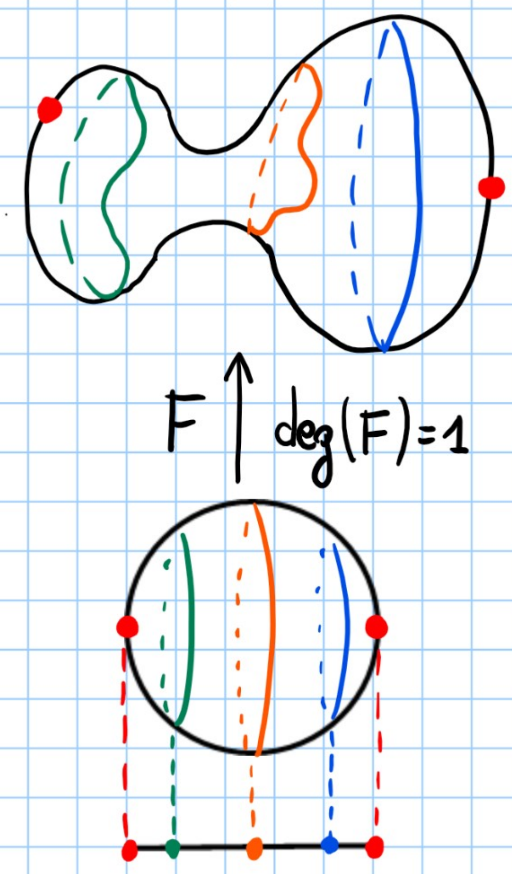
**Thm 5 (Pu)** $\sigma(g) \leq \pi/2\ \forall g \in \mathcal{R}(P^2)$ and $\sigma(g) = \pi/2$ iff $g = g_0$.

**Thm 6 (Bavard)** $\sigma(g) \leq \frac{\pi}{2\sqrt{2}}\ \forall g \in \mathcal{R}(K^2)$. $\frac{\pi}{2\sqrt{2}}$ is the optimal constant but the maximizing metric is not smooth!

**Thm 7 (Gromov)** $\exists\, C > 0$ s.t. $\sigma(g) \leq C \cdot \frac{(\log k)^2}{k}\ \forall g \in \mathcal{R}(\Sigma_k) \cup \mathcal{R}(\tilde{\Sigma}_k)$.

# 3. Systoles on $S^2$ ?

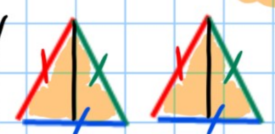
**Def** A weak systole of $g \in \mathcal{R}(\Sigma)$ is a (non-constant), periodic geodesic whose length is minimal (among all non-constant periodic geodesic). We denote by $\widetilde{sys}(g)$ its length and by $\tilde{\sigma}(g) := \frac{\widetilde{sys}(g)^2}{\mathrm{area}_g(\Sigma)}$ the weak systolic ratio. (Note: $\tilde{\sigma}(g) \leq \sigma(g)$)



Birkhoff's theorem

**Thm 8 (Birkhoff)** Every $g \in \mathcal{R}(S^2)$ has a non-constant periodic geodesic.

**Thm 9 (Croke, Rotman)** There exists $C < 2\sqrt{8}$ such that $\tilde{\sigma}(g) \leq C \ \forall g \in \mathcal{R}(S^2)$

**Conjecture** Optimal $C = 2\sqrt{3} = \tilde{\sigma}\Big($  $\Big)$.

**Remark** $\tilde{\sigma}(g_0) = \frac{(2\pi)^2}{4\pi} = \pi < 2\sqrt{3} \implies g_0$ is not a global maximizer.
Is $g_0$ at least a local maximizer? Yes, but it is not alone...

**Def** $g \in \mathcal{R}(\Sigma)$ is called Zoll if all geodesics of $g$ are periodic and with same length. Denote $\mathcal{Z}(\Sigma)$ the set of Zoll metrics.

**3 Facts**:
i) $\mathcal{Z}(\Sigma) \neq \emptyset \implies \Sigma = S^2, P^2$.
ii) $\mathcal{Z}(P^2) = \{g_0\}$ (Green).
iii) $\mathcal{Z}(S^2)$ infinite dimensional!

$h : [-1,1] \to (-1,1)$ odd, $h(-1) = 0 = h(1)$

Surfaces of revolution $g = (1 + h(\cos\theta))^2 d\theta^2 + \sin^2\theta \, d\varphi^2 \in \mathcal{Z}(S^2)$ (Otto Zoll)
Let $\psi : S^2 \to \mathbb{R}$ be odd $(\psi(-x) = -\psi(x))$. Then:
$\exists\, g_s = f_s^2 \cdot g_0 \in \mathcal{Z}(S^2) \ \forall s \in (-\varepsilon, \varepsilon)$ with $f_0 = 1$, $\dot{f}_0 = \psi$ (Guillemin)

**Thm 10 (Weinstein)** $\tilde{\sigma}(g) = \pi \ \forall g \in \mathcal{Z}(S^2)$.

**Thm 11 (ABHS)** Let $g_* \in \mathcal{Z}(S^2)$. If $g \overset{C^3}{\sim} g_*$, then $\tilde{\sigma}(g) \leq \pi$ and $\tilde{\sigma}(g) = \pi \iff g \in \mathcal{Z}(S^2)$.

ABHS = Abbondandolo, Bramham, Hryniewicz, Salomão

**Thm 12 (ABHS)** Let $g \in \mathcal{R}(S^2)$ coming from a surface of revolution in $\mathbb{R}^3$. Then: $\tilde{\sigma}(g) \leq \pi$ and $\tilde{\sigma}(g) = \pi \iff g \in \mathcal{Z}(S^2)$.

Proof of Thm 10-11-12 uses symplectic geometry (next talk!).