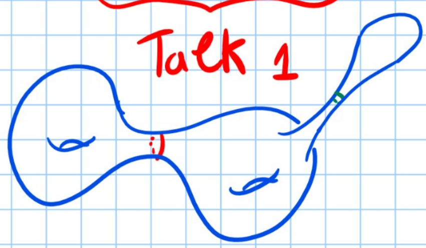
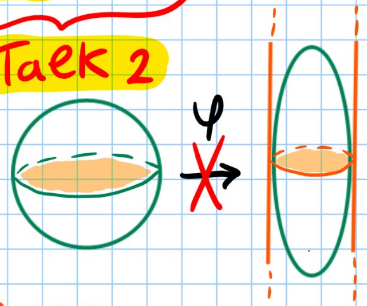
FIRST STEPS INTO THE WORLD OF SYSTOLIC INEQUALITIES

From $\underbrace{\text{Riemannian}}_{\text{Talk 1}}$ to $\underbrace{\text{Symplectic}}_{\text{Talk 2}}$ Geometry





GABRIELE BENEDETTI
UNIVERSITY OF HEIDELBERG

DDTG SEMINAR 27/11/2020

1. Hamiltonian Systems

Phase space $\underset{(q,p)}{\mathbb{R}^n \times \mathbb{R}^n} \cong \underset{z}{\mathbb{R}^{2n}}$. Every $H:\mathbb{R}^{2n}\to\mathbb{R}$ yields solutions $t\mapsto (q(t),p(t))$ of

$$\boxed{\begin{cases}\dot q(t) = +\frac{\partial H}{\partial p}(q(t),p(t))\\ \dot p(t) = -\frac{\partial H}{\partial q}(q(t),p(t))\end{cases}}$$

$\left(\iff \dot z(t) = X_H(z(t)),\ X_H = \begin{pmatrix}+\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial q}\end{pmatrix}\right)$. Coordinate-free expression of X_H via canonical symplectic form $\omega_0 = \sum_{i=1}^n dp_i\wedge dq_i$:

$\boxed{\omega_0(X_H,\cdot) = -dH}$ "X_H is minus the symp. gradient of H"

Let ϕ_H^t be the flow of X_H. Conservation of energy: $\frac{d}{dt}(H\circ\phi_H^t) = dH\cdot X_H = -\omega_0(X_H,X_H) = 0$.

Example $H_a(q,p) = \frac{1}{2}\sum_{i=1}^n \frac{q_i^2+p_i^2}{a_i}$, $0<a_1\le\ldots\le a_n$ (n harmonic oscillators with frequencies $\frac{a_i}{\pi}$)

Remark More generally, Hamiltonian systems defined on manifolds M with symplectic form ω (non-degenerate two-form, $d\omega=0$).

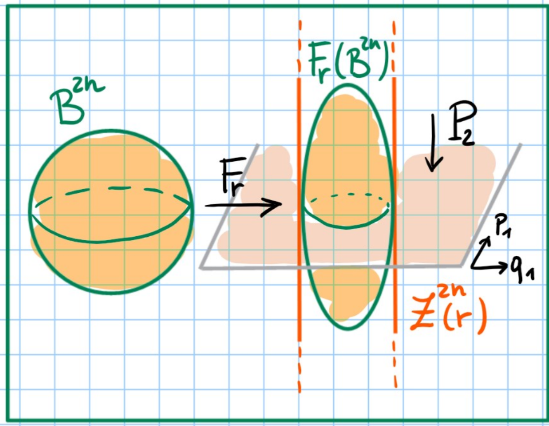
Example $M = T^*\Sigma$, $\omega = \sum_{i=1}^n dp_i\wedge dq_i$ (q local coordinates on Σ, p associated momenta)
Every $g\in \mathcal{R}(\Sigma)$ yield: i) $T\Sigma \xrightarrow{b} T^*\Sigma$ ii) $H_g: T^*\Sigma\to\mathbb{R}$, $H_g(q,p) = \frac{1}{2}|p|_g^2$.
Then: $z:\mathbb{R}\to T^*\Sigma$ flow-line of $\phi_{H_g} \iff z = b(\dot\gamma)$, where $\gamma:\mathbb{R}\to\Sigma$ geodesic.

2. Symplectic Geometry

$G := \{\varphi:\mathbb{R}^{2n}\to\mathbb{R}^{2n} \mid \varphi \text{ diffeo}, \varphi^*\omega_0 = \omega_0\}$ (group of symplectomorphisms)
$\omega_0^n = n!\,\mathrm{vol}_{2n} \Rightarrow G\subset G' := \{F:\mathbb{R}^{2n}\to\mathbb{R}^{2n} \mid F \text{ diffeo}, F^*\mathrm{vol}_{2n} = \mathrm{vol}_{2n}\}$

Question: What's the difference between G and G' for $n>1$? Compare how they act on $B^{2n} = B^{2n}(1)$, where $B^{2n}(r)\subseteq\mathbb{R}^{2n}$ is ball of radius r.

Remark $\forall r>0,\ \exists F_r \in G',\ F_r(B^{2n}) \subseteq Z^{2n}(r)$.
$Z^{2n}(r) := \{ q_1^2 + p_1^2 \le r^2 \}$.



Thm 1 (Gromov's non-squeezing) $\forall \varphi \in G$
$$\mathrm{vol}_2\left[P_2(\varphi(B^{2n}))\right] \geq \pi = \mathrm{vol}_2(B^2).$$
$(\Rightarrow \nexists\, \varphi \in G,\ \varphi(B^{2n}) \subseteq Z^{2n}(r)$ for $r<1)$.

Question: Let $1<k<n$. What can be said of $\mathrm{vol}_{2k}\left[P_{2k}(\varphi(B^{2n}))\right]$ for $\varphi \in G$?

Thm 2 (Abbondandolo-Matveyev) $\forall \varepsilon>0\ \exists \varphi_\varepsilon \in G,\ \mathrm{vol}_{2k}\left[P_{2k}(\varphi_\varepsilon(B^{2n}))\right] < \varepsilon$.

Thm3 (Abbondandolo-B.) Let $\varphi_0 \in G$, φ linear. If $\varphi \in G$, $\varphi \overset{C^3}{\sim} \varphi_0$, then
$$\mathrm{vol}_{2k}\left[P_{2k}(\varphi(B^{2n}))\right] \geq \frac{\pi^k}{k!} = \mathrm{vol}_{2k}(B^{2k}).$$

Question: where does the boundary between squeezing and non-squeezing lie?

Conjecture 1 If $\varphi \in G$, $\varphi(B^{2n})$ is convex, then $\mathrm{vol}_{2k}\left[P_{2k}(\varphi(B^{2n}))\right] \geq \frac{\pi^k}{k!}$.

3. Symplectic Measurements

Def $c: \{A \subseteq \mathbb{R}^{2n}\} \to [0,\infty]$ is a symplectic capacity, if

i) $\forall A_1 \subseteq A_2 \subseteq \mathbb{R}^{2n}$: $c(A_1) \leq c(A_2)$

ii) $\forall A \subseteq \mathbb{R}^{2n}, \forall \varphi \in G$: $c(\varphi(A)) = c(A)$

iii) $\forall A \subseteq \mathbb{R}^{2n}, \forall r>0$: $c(rA) = r^2 c(A)$

iv) $c(B^{2n}(1)) = \pi = c(Z^{2n}(1))$.

$\exists$ capacity $c \Rightarrow$ Thm 1 $\Rightarrow c_B$ and c_Z are capacities and $c_B \leq c \leq c_Z$ $\forall$ capacities c,
where $c_B(A) := \sup\{\pi r^2 \mid \exists \varphi \in G,\ \varphi(B^{2n}(r)) \subseteq A\}$,
$c_Z(A) := \inf\{\pi r^2 \mid \exists \varphi \in G,\ \varphi(A) \subseteq Z^{2n}(r)\}$.

Example $E(a) := \{H_a \leq \frac{1}{2}\}$ ellipsoid: $B^{2n}(\sqrt{a_1}) \subseteq E(a) \subseteq Z^{2n}(\sqrt{a_1}) \Rightarrow c(E(a)) = \pi a_1$.

Conjecture 2 All capacities coincide on convex bodies A (compact, $\mathring{A} \neq \emptyset$).

Remark (Ostrover): True, if $\mu \cdot A = A$ $\forall \mu \in \mathbb{C}, |\mu| = 1$. (identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$)
Computing capacities of subset A is hard, let's estimate them using the volume of A.

Example
$$\frac{c(E(a))^n}{n!\,\mathrm{vol}_{2n}(E(a))} = \frac{(\pi a_1)^n}{n! \frac{\pi^n}{n!} a_1 \cdot \ldots \cdot a_n} = \frac{a_1^n}{a_1 \cdot \ldots \cdot a_n} \leq 1 \quad (\ldots = 1 \Leftrightarrow a_1 = \ldots = a_n)$$

Conjecture 3 (Viterbo) Let c be a capacity. For all convex bodies A, there holds:
$$\rho_c(A) := \frac{c(A)^n}{n!\,\mathrm{Vol}_{2n}(A)} \leq 1 \quad \text{and} \quad \left[\rho_c(A) = 1 \iff A \overset{\text{symp}}{\cong} B^{2n}(r)\right]$$

Remark Conjecture 3 holds with $c = c_B$: Conjecture 2 $\Rightarrow$ Conjecture 3 for all c.

Thm 4 (Artstein-Avidan – Ostrover – Milman) $\exists C \geq 1$ ind. of c and n:
$\rho_c(A) \leq C^n$ for all convex bodies A.
(idea: approximate A with ellipsoids)

4. Capacities from periodic orbits (p.o.)

Conservation of energy: $\forall h \in \mathbb{R}$, $S := \{H = h\} \subset \mathbb{R}^{2n}$ invariant by ϕ_H.

Question Let S be regular and compact. Does it exist a p.o. of ϕ_H on S?

Remark Up to time reparametrization, $\phi_H|_S$ depends only on S and not on H!

(Maupertuis): Let λ be 1-form, $d\lambda = \omega_0$. z p.o. on $S \iff z$ critical point of $w \mapsto \mathbb{A}(w) = \int_w \lambda$, $w : \mathbb{R}/_{T\mathbb{Z}} \to S$ immersed.

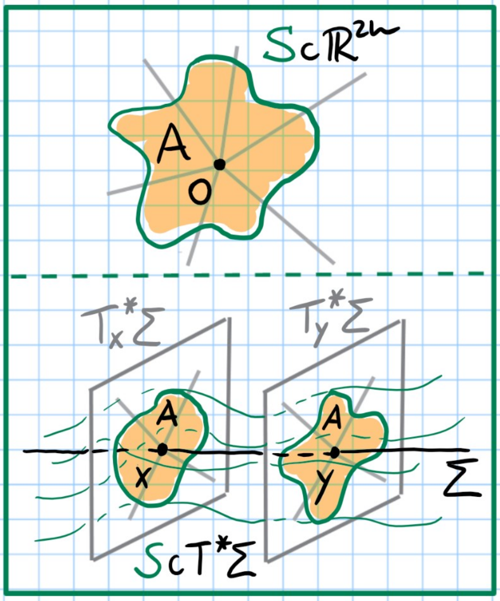
Difficulty No a priori bound on period of z. $\exists S \subset \mathbb{R}^{2n}$ without p.o.!

Definition S is contact, if $\lambda(X_H|_S) > 0$. $\widetilde{sys}(S) := \min\{\mathbb{A}(z) \mid z \text{ p.o. on } S\} \in (0, \infty]$

Thm 5 (Viterbo) If $S \subset \mathbb{R}^{2n}$ is of contact type, then $\exists z$ periodic orbit on S.

Examples i) $\lambda = \frac{1}{2}\sum_{i=1}^{n}(p_i dq_i - q_i dp_i)$ in $\mathbb{R}^{2n}$.
$S = \partial A$ is contact, if A starshaped about 0.
E.g. A is convex body, $0 \in \mathring{A}$.

ii) $\lambda = \sum_{i=1}^{n} p_i dq_i$ in $T^*\Sigma$. $S \subset \partial A$ is contact,
Thm 5 Known for these $\to$ if A fibrewise starshape about 0-section.
E.g. $A = \{H_g \leq 1/2\}$ for $g \in R(\Sigma)$ and
$$\mathbb{A}(b(\gamma)) = \ell_g(\gamma)$$
$$\widetilde{sys}(\{H_g = 1/2\}) = \widetilde{sys}(g)$$



Thm 6 (Hofer-Zehnder) $\exists$ capacity $\tilde{c}$ such that $\tilde{c}(A) = \widetilde{sys}(\partial A)$, if A convex body.

Conjecture 3 for $\tilde{c}$ $\Rightarrow \begin{cases} \text{Conjecture 1} \\ \text{Mahler's conjecture: } \mathrm{vol}_n(\underbrace{K}_{\text{unit ball of a norm in } \mathbb{R}^n}) \cdot \mathrm{vol}_n(\underbrace{K^\circ}_{\text{unit ball of dual norm}}) \geq 4^n/n! \end{cases}$

Thm 7 (Abbondandolo - B.) Let $A \subset \mathbb{R}^{2n}$ convex body, $A \overset{C^3}{\sim} B^{2n}$. Then:

$$\tilde{\sigma}(A) := \rho_{\tilde{c}}(A) \leq 1 \quad \text{and} \quad \tilde{\sigma}(A) = 1 \iff A \overset{\text{symp}}{\cong} B^{2n}(r).$$

Thm 7 says that Conj. 3 for $\tilde{c}$ holds locally around B^{2n}. This implies Thm 3.

5. Local systolic inequality for contact manifolds

Def A contact S is called Zoll if all orbits on S are periodic and with same action $\mathbb{A}$.

Example $S = \partial B^{2n}(r)$ is Zoll. $g \in R(\Sigma)$ is Zoll $\iff \{H_g = 1/2\} \subset T^*\Sigma$ is Zoll.

Def Let S be contact. We define $\mathrm{vol}(S) := \int_S \lambda \wedge (d\lambda)^{n-1}$ and $\tilde{\sigma}(S) := \dfrac{\widetilde{\mathrm{sys}}(S)}{\mathrm{vol}(S)}$.

Example i) If $S = \partial A$, $A \subset \mathbb{R}^{2n}$, $\mathrm{vol}(S) = \int_A \omega_0^n = n!\, \mathrm{vol}_{2n}(A)$.

ii) If $S = \{H_g = 1/2\} \subset T^*\Sigma$, $\mathrm{vol}(S) = \underbrace{a_n}_{n!\,\mathrm{vol}_n(B^n)} \cdot \mathrm{vol}_g(\Sigma)$, $\tilde{\sigma}(S) = \dfrac{1}{a_n}\tilde{\sigma}(g)$.

Thm 8 (Abbondandolo - B.) Let S_0 be contact and Zoll. Let S be contact, $S \overset{C^3}{\sim} S_0$:

$$\tilde{\sigma}(S) \leq \tilde{\sigma}(S_0) \quad \text{and} \quad \tilde{\sigma}(S) = \tilde{\sigma}(S_0) \iff S \text{ is Zoll}.$$

Thm 8 $\Rightarrow$ Thm 7 above and Thm 11 from last time (local systolic inequality on S^2) $S^n, P^n, \mathbb{C}P^n$

Sketch of proof of Thm 8

Step 1 Let S be contact. Define $\alpha := \lambda|_S$ and $R_\alpha = \dfrac{X_H|_S}{\alpha(X_H|_S)}$. Then:

$\alpha(R_\alpha) = 1$, $d\alpha(R_\alpha, \cdot) = 0$. $\mathbb{A}(z) = T(z)$ period

Step 2 If S_0 is Zoll with form α_0. Let $\pi: S_0 \to N := S_0/\phi_H^t$ orbit space.

N is closed manifold, π is a principal S^1-bundle,

α_0 is a connection form for π with curvature K on N: $d\alpha_0 = \pi^* K$.

S_0 contact $\Rightarrow$ K symplectic form on N. Ex: $\pi: \partial B^{2n} \to \mathbb{C}P^n$ Hopf fibration, K Fubini-Study

Step 3 Let $S \overset{C^3}{\sim} S_0$. Up to a diffeo $S = S_0$ and we consider both α and α_0 on S_0.

Find a normal form for α: $\alpha = (f \circ \pi) \cdot \alpha_0 + \text{reminder}$, $f: N \to (0, \infty)$.

For simplicity reminder $= 0$ below.

Step 4 Which π-fibers are orbits of R_α? Need to check $0 = d\alpha(R_{\alpha_0}, \cdot)$

$$d\alpha(R_{\alpha_0}, \cdot) = \pi^*(df)\wedge\alpha_0(R_{\alpha_0}, \cdot) + \pi^*(fK)(R_{\alpha_0}, \cdot) = \pi^* df.$$

$\Rightarrow$ If $x \in N$ is critical point of f, then $\pi^{-1}(x)$ p.o. of R_α and

$$\mathcal{A}(\pi^{-1}(x)) = f(x)\cdot \widetilde{sys}(S_0, \alpha_0) \Longrightarrow \widetilde{sys}(S_0,\alpha) \leq \min f \cdot \widetilde{sys}(S_0, \alpha_0).$$

Step 5 $\displaystyle vol(S_0, \alpha) = \int_{S_0} (f\circ\pi)\alpha_0 \wedge d[(f\circ\pi)\alpha_0]^{n-1} = \int_N f^n K.$

Step 6 Use the inequality between minimum and arithmetic mean

$$vol(S_0,\alpha) = \int_N f^n K \geq (\min f)^n \cdot \int_N K \geq \frac{\widetilde{sys}(S_0,\alpha)^n}{\widetilde{sys}(S_0,\alpha_0)^n} \cdot vol(S_0, \alpha_0)$$

Rearranging: $\tilde{\sigma}(S_0, \alpha_0) \geq \tilde{\sigma}(S_0, \alpha)$.

$\tilde{\sigma}(S_0, \alpha_0) = \tilde{\sigma}(S_0, \alpha) \Rightarrow f$ is constant $\Rightarrow (S_0, \alpha)$ is Zoll. **The End.**