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Topological complexity - introduction and perspectives

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Overview

First part: Introduction to topological complexity

- Definition and basic properties
- Computations of topological complexity
- Collision-free motion planning

Second part: Recent topics related to topological complexity

- Topology and robot kinematics
- Parametrized topological complexity
- Geodesic complexity
- Spherical complexities

Definition and basic properties

Topology and robot motion planning

Real-world situation

A robot is supposed to move autonomously from one location to another in its workspace (e.g. warehouse, grid network, ...).

Topological motion planning problem

Let X be a path-connected topological space. Given $x, y \in X$, find a path $\gamma \in PX = C^0([0,1], X)$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Definition

Let X be a top. space, $A \subset X \times X$. A *motion planner over A* is a map $s : A \to PX$, such that $(s(x,y))(0) = x$, $(s(x,y))(1) = y$, for all $(x,y) \in A$, i.e. a section over A of the fibration

$$\pi : PX \to X \times X, \qquad \gamma \mapsto (\gamma(0), \gamma(1)).$$

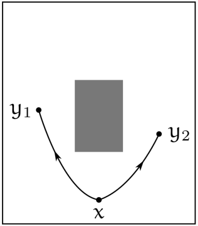
For a robot to move autonomously in X, we need a motion planner over $X \times X$.

Existence of motion planners

Q: How "simple" can a motion planner over $X \times X$ be chosen?

Theorem (Farber, '03)

Let X be a path-conn. top. space. There exists a continuous motion planner over $X \times X$ if and only if X is contractible.



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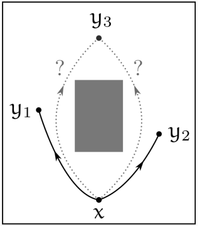


Proof of "$\Rightarrow$".
Let $s : X \times X \to PX$ be a continuous motion planner. Fix $x_0 \in X$. Define $H : X \times [0,1] \to X$, $H(x,t) = (s(x_0,x))(t)$. Then H is a homotopy contracting X onto $\{x_0\}$. $\square$

Topological complexity

We want robots to move *predictably*, so we want motion planners to be continuous on large subsets of $X \times X$ and only have few "jumps" in the path assigments.

Idea: Search for the lowest number of "jumps" of a motion planner that is necessary by the topology of the space.

Definition (Farber ’03)

Let X be a path-connected top. space. The *topological complexity of X* is given by $\mathrm{TC}(X) \in \mathbb{N} \cup \{+\infty\}$,

$$\mathrm{TC}(X) := \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = X \times X \text{ open cover, s.t.}$$
$$\forall j \; \exists s_j : U_j \to PX \text{ cont. motion planner}\Big\}.$$

Basic properties of topological complexity

Theorem

If X is a Euclidean neighborhood retract (e.g. a compact manifold, a finite CW complex), then

$$\mathrm{TC}(X) := \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigsqcup_{j=1}^{n} A_j = X \times X,\ \textit{s.t. } A_j \textit{ locally compact}, \\ \forall j\ \exists s_j : A_j \to PX \textit{ cont. motion planner} \Big\}$$

Theorem $\mathrm{TC}(X)$ is a homotopy invariant, i.e. if $X \simeq Y$, then $\mathrm{TC}(X) = \mathrm{TC}(Y)$.

Theorem If X is an r-connected polyhedron, then

$$\mathrm{TC}(X) \leq \frac{2\dim X + 1}{r+1} + 1.$$

Remark $\mathrm{TC}(X) = 1 \;\Leftrightarrow\; X$ is contractible.

Approach to computing TC

Problem In general, $\mathrm{TC}(X)$ is hard to compute.

Usual strategy to compute it:

- Find a good **upper bound** for $\mathrm{TC}(X)$, e.g. by one of the following:
  - Apply the dim.-connectivity upper bound from above.
  - Find an explicit open cover with k sets admitting continuous motion planners. Then $\mathrm{TC}(X) \leq k$.
- Find a good **lower bound** for $\mathrm{TC}(X)$, e.g. by
  - Cohomology methods ($\longrightarrow$ later)
  - Comparison to other fibrations ($\longrightarrow$ later)

Try to find bounds with "lower bound $=$ upper bound" to compute $\mathrm{TC}(X)$.

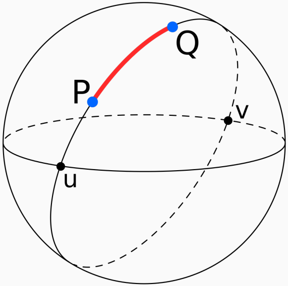
Example It was only shown in 2016 by Cohen-Vandembroucq that TC(Klein bottle) $= 5$.

Example: The topological complexity of spheres (1)

Since S^n is not contractible, $\mathrm{TC}(S^n) \geq 2$ for all $n \in \mathbb{N}$. Define a motion planner as follows:

$$A_1 = \{(x, y) \in S^n \times S^n \mid y \neq -x\}, \quad s_1 : A_1 \to PS^n,$$

$$s_1(x, y) = \text{param. minimal great circle segment from } x \text{ to } y.$$



Example: The topological complexity of spheres (2)

For n odd, fix a nowhere vanishing vector field X on S^n, put

$$A_2 = \{(x, -x) \in S^n \times S^n \mid x \in S^n\}, \quad s_2 : A_2 \to PS^n,$$
$$s_2(x, -x) = \text{parametrized semicircle tangent to } X(x).$$

Then $\;A_1 \sqcup A_2 = S^n \times S^n \quad \Rightarrow \quad \mathrm{TC}(S^n) = 2.$

For n even, fix a vector field X on S^n vanishing at a unique point $x_0 \in S^n$. Put

$$A_2 = \{(x, -x) \mid x \in S^n \setminus \{x_0\}\},$$

define $s_2 : U_2 \to PS^n$ as the odd case. Put

$A_3 = \{(x_0, -x_0)\},\; s_3(x_0, -x_0) =$ some semicircle from x_0 to $-x_0$.

$$A_1 \sqcup A_2 \sqcup A_3 = S^n \times S^n \quad \Rightarrow \quad \mathrm{TC}(S^n) \leq 3.$$

Hence, $\mathrm{TC}(S^n) \in \{2, 3\}$ for even n. ($\to$ *later)*

Why study topological complexity?

- Practical applications in the design of automated mechanical systems
- Connections with other problems from geometry, e.g. $\mathrm{TC}(\mathbb{R}P^n)$ and existence of immersions $\mathbb{R}P^n \to \mathbb{R}^k$ ($\longrightarrow$ details later)
- interesting in its own right as a homotopy invariant, e.g. **Problem** Let π be a discrete group. How can we express $\mathrm{TC}(K(\pi, 1))$ as an algebraic invariant of π?

A similar invariant: Lusternik-Schnirelmann category

Definition Let X be a top. space. The *Lusternik-Schnirelmann category of X* is given by $\operatorname{cat}(X) \in \mathbb{N} \cup \{+\infty\}$,

$$\operatorname{cat}(X) := \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = X \text{ open cover, s.t.} \\ \operatorname{incl}_{U_j} : U_j \hookrightarrow X \text{ is nullhomotopic } \forall j \Big\}.$$

Motivated by connection to critical point theory:

Theorem (Lusternik-Schnirelmann '34, Palais '65)
Let M be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on M. Then

$$\#\operatorname{Crit} f \geq \operatorname{cat}(M).$$

$\longrightarrow$ book by Cornea/Lupton/Oprea/Tanré (2003)

A more general notion: sectional category

Definition (A. Schwarz, '61)

Let $p : E \to B$ be a fibration. The *sectional category* or *Schwarz genus* of p is given by

$$\operatorname{secat}(p) = \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = B \text{ open cover}, \\ s_j : U_j \xrightarrow{C^0} E,\ \ p \circ s_j = \mathrm{incl}_{U_j} \ \ \forall j \Big\}.$$

Special cases:

$$\mathrm{TC}(X) = \operatorname{secat}\Big(\pi : PX \to X \times X,\ \gamma \mapsto (\gamma(0), \gamma(1))\Big),$$

$$\operatorname{cat}(X) = \operatorname{secat}\Big(p : P_{x_0}X = \{\gamma \in PX \mid \gamma(0) = x_0\} \to X,\ \gamma \mapsto \gamma(1)\Big)$$

Relations between cat and TC:

$$\operatorname{cat}(X) \leq \mathrm{TC}(X) \leq \operatorname{cat}(X \times X), \qquad \mathrm{TC}(G) = \operatorname{cat}(G) \text{ if } G \text{ top. group.}$$

Imagine a robot is moving to a desired location in a warehouse. What if the robot needs to pick up items on its way?

Definition (Rudyak, '10)

Let X be a path-conn. top. space, $n \in \mathbb{N}$ with $n \geq 2$, consider

$$\pi_n : PX \to X^n, \quad \gamma \mapsto \left(\gamma(0), \gamma\left(\tfrac{1}{n-1}\right), \ldots, \gamma\left(\tfrac{n-2}{n-1}\right), \gamma(1)\right).$$

The *n-th topological complexity* is defined by

$$\mathrm{TC}_n(X) := \mathrm{secat}(\pi_n : PX \to X^n).$$

A characterization of sectional category

For fibrations $q_1 : E_1 \to B$ and $q_2 : E_2 \to B$ let $q_1 *_f q_2 : E_1 *_f E_2 \to B$ denote the *fiberwise join* of q_1 and q_2, i.e. $q_1 * q_2$ is a fibration with fiber over $b \in B$ given by

$$(E_1 *_f E_2)_b = (E_1)_b * (E_2)_b,$$

where $*$ denotes the join of two top. spaces.
Given a fibration $p : E \to B$, define $p_n : E_n \to B$, $n \in \mathbb{N}$, by

$$p_1 := p, \quad p_n := p_{n-1} *_f p : E_n := E_{n-1} *_f E \to B \;\; \forall n > 1.$$

Theorem (Schwarz '61)

$$\mathrm{secat}(p : E \to B) = \inf\{n \in \mathbb{N} \mid \exists s : B \overset{C^0}{\to} E_n \textit{ with } p_n \circ s = \mathrm{id}_B\}.$$

$\longrightarrow$ Study secat(p) by means of obstruction theory and other tools from homotopy theory.

A lower bound by cohomology methods

Given a commutative ring R, a topological space Y and an ideal $I \subset H^*(Y; R)$, let

$$\mathrm{cl}(I) := \sup\{n \mid \exists u_1, \ldots, u_n \in I \cap \widetilde{H}^*(Y; R),\ \text{s.t. } u_1 \cup \cdots \cup u_n \neq 0\}.$$

Theorem (Schwarz, '61)

Let $p : E \to B$ be a fibration, R be a commutative ring. Then

$$\mathrm{secat}(p) \geq \mathrm{cl}\left(\ker\left[p^* : H^*(B; R) \to H^*(E; R)\right]\right) + 1.$$

Put $\quad Z_R(X) := \ker\left[\Delta^* : H^*(X \times X; R) \to H^*(X; R)\right]$,

where $\Delta(x) = (x, x)$. We call elements of $Z_R(X)$ *zero-divisors.*

A lower bound for topological complexity

Corollary (Farber, 2003) $\mathrm{TC}(X) \geq \mathrm{cl}(Z_R(X)) + 1$.

Proof.
$\overset{\text{Theorem}}{\Rightarrow} \mathrm{TC}(X) \geq \mathrm{cl}\left(\ker[\pi^* : H^*(X \times X; R) \to H^*(PX; R)]\right) + 1$.

Let $f : X \to PX$, $(f(x))(t) = x$ for all $t \in [0,1]$, $x \in X$ (inclusion of constant paths). Then the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\Delta"'] & & PX \arrow[dl, "\pi"] \\ & X \times X & \end{tikzcd}$$

But f is a homotopy equivalence with homotopy inverse $\mathrm{ev}_0 : PX \to X$, $\mathrm{ev}_0(\gamma) = \gamma(0)$. $\square$

Note If $\sigma \in H^*(X; R)$, then $\bar{\sigma} := 1 \times \sigma - \sigma \times 1 \in Z_R(X)$.

Improved lower bound using weights of cohomology classes

(Farber-Grant '07, following Fadell-Husseini '92, Rudyak '99)

We can sharpen the lower bound by assigning *weights* to cohomology classes,

$$\mathrm{wgt} : \widetilde{H}^*(X \times X; R) \to \mathbb{N}_0,$$

with

$$\mathrm{wgt}(u) \geq 1 \quad \Leftrightarrow \quad u \in Z_R(X),$$

such that if $u_1, \ldots, u_k \in \widetilde{H}^*(X \times X; R)$ with $u_1 \cup u_2 \cup \cdots \cup u_k \neq 0$, then

$$\mathrm{TC}(X) \geq \sum_{i=1}^{k} \mathrm{wgt}(u_i) + 1.$$

Example: The topological complexity of spheres (3)

Have seen that $\mathrm{TC}(S^n) \in \{2, 3\}$ if n is even.

Let $[S^n] \in H^n(S^n; \mathbb{R})$ be the fundamental class, put

$$\sigma := 1 \times [S^n] - [S^n] \times 1 \in Z_{\mathbb{R}}(S^n).$$

Then

$$\sigma \cup \sigma = -2 \cdot [S^n] \times [S^n] \neq 0 \quad \Rightarrow \quad \mathrm{cl}(Z_{\mathbb{R}}(S^n)) \geq 2.$$

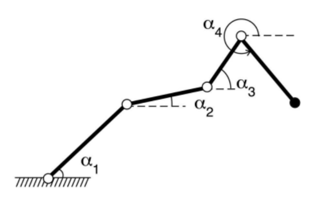
By previous corollary, $\mathrm{TC}(S^n) \geq 3$. Hence,

$$\mathrm{TC}(S^n) = 3 \quad \text{if } n \text{ is even.}$$

Computations of topological complexities

Robot arms (1)

Consider a robot arm moving in a plane, fixed at a base point, but otherwise freely moving, consisting of m bars $L_1, \ldots, L_m$, s.t. L_i is connected to L_{i+1} by a flexible join for each $i \in \{1, 2, \ldots, m-1\}$.



(Picture stolen from Farber, *Topological complexity of motion planning*)

Its space of possible positions is given by $T^m = (S^1)^m$.

Robot arms (2)

Theorem If X, Y are ENRs, then $\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$.

Iterating this result, we obtain

$$\mathrm{TC}(T^m) \leq m \cdot \mathrm{TC}(S^1) - (m - 1) = m + 1.$$

The cohomology lower bound shows that $\mathrm{TC}(T^m) \geq m + 1$, hence $\mathrm{TC}(T^m) = m + 1$.

Assume now the same linkage is moving in three-dimensional space. Its space of possible positions is

$$M = (S^2)^m.$$

Here, $\mathrm{TC}(M) \leq m \cdot \mathrm{TC}(S^2) - m + 1 = 2m + 1$. Cohomology lower bound yields $\mathrm{TC}(M) = 2m + 1$.

Grid networks

Consider an automatically guided vehicle moving along a wire network that is fixed on the ground.

Theorem

If Γ is a finite graph (seen as a one-dim. CW complex), then

$$\mathrm{TC}(\Gamma) = \begin{cases} 1 & \textit{if } \pi_1(\Gamma) = \{1\}, \\ 2 & \textit{if } \pi_1(\Gamma) \cong \mathbb{Z}, \\ 3 & \textit{else.} \end{cases}$$

Proof If $\pi_1(\Gamma) = \{1\}$, then Γ is contractible $\Rightarrow \mathrm{TC}(\Gamma) = 1$.
If $\pi_1(\Gamma) \cong \mathbb{Z}$, then Γ has the homotopy type of S^1. $\Rightarrow \mathrm{TC}(S^1) = 2$.
If $\pi_1(\Gamma) \notin \{\mathbb{Z}, \{1\}\}$, then $\pi_1(\Gamma) \cong \mathbb{Z}^{*n} = \mathbb{Z} * \cdots * \mathbb{Z}$ for some $n \geq 2$ and $\Gamma \simeq \bigvee_{i=1}^{n} S^1$. Then $\mathrm{cl}(H^*(\Gamma; \mathbb{Z})) \geq 2$ and one derives that $\mathrm{cl}(Z_{\mathbb{Z}}(\Gamma)) \geq 2$. $\Rightarrow \mathrm{TC}(\Gamma) \geq 3$.
Since $\mathrm{TC}(\Gamma) \leq 2\dim \Gamma + 1 = 3$, the claim follows. $\square$

Real projective spaces (1)

Consider a metallic bar in 3-dim. space. whose center is fixed at a revolving joint. Its space of positions corresponds to $\mathbb{R}P^3$.

Question What is $\mathrm{TC}(\mathbb{R}P^n)$?
The lower and upper bounds from above only show

$$n+1 \leq \mathrm{TC}(\mathbb{R}P^n) \leq 2n+1.$$

Theorem (Farber-Tabachnikov-Yuzvinsky '03)

$$\mathrm{TC}(\mathbb{R}P^n) = \begin{cases} I_n & n \in \{1,3,7\}, \\ I_n + 1 & \textit{else,} \end{cases}$$

where $I_n = \inf\{k \in \mathbb{N} \mid \exists \text{ immersion } \mathbb{R}P^n \to \mathbb{R}^k\}$.

"$\leq$" is shown constructing explicit local motion planners via an immersion $\mathbb{R}P^n \to \mathbb{R}^k$.

Real projective spaces (2)

What about "$\geq$"? (only method of proof in what follows)

Lemma
Let $q : \tilde{X} \to X$ be a regular G-covering, let $\tilde{X} \times_G \tilde{X} := (\tilde{X} \times \tilde{X})/\sim$, where $(x_1, x_2) \sim (gx_1, gx_2)$ for all $g \in G$, $x_1, x_2 \in \tilde{X}$, $p : \tilde{X} \times_G \tilde{X} \to X \times X$ be induced by $q \times q$. Then

$$\mathrm{TC}(X) \geq \mathrm{secat}(p : \tilde{X} \times_G \tilde{X} \to X \times X).$$

Proof Let $f : PX \to \tilde{X} \times_G \tilde{X}$, $f(\gamma) = [(\tilde{\gamma}(0), \tilde{\gamma}(1))]$, where $\tilde{\gamma}$ a lift of γ. Then $p \circ f = \pi$.

$\Rightarrow$ If $s : U \to PX$ is a local section of π, then $f \circ s$ is a local section of p. $\square$

Apply lemma to two-fold covering $q : S^n \to \mathbb{R}P^n$ to obtain

$$\mathrm{TC}(\mathbb{R}P^n) \geq \mathrm{secat}(p : S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{R}P^n \times \mathbb{R}P^n).$$

Real projective spaces (3)

Recall that

$$\begin{aligned}&\mathrm{secat}(p)\\&=\inf\{k\in\mathbb{N}\mid\exists\text{ cont. section of }p^{*k}:(S^n\times_{\mathbb{Z}_2}S^n)^{*k}\to\mathbb{R}P^n\times\mathbb{R}P^n\},\end{aligned}$$

where p^{*k} is the k-fold fiberwise join of p with itself.

Let $\xi\to\mathbb{R}P^n$ be the tautological line bundle. One shows that for each $k\in\mathbb{N}$, p^{*k} is the unit sphere bundle of

$$k(\xi\otimes\xi)=\bigoplus_{i=1}^{k}(\xi\otimes\xi)\to\mathbb{R}P^n\times\mathbb{R}P^n.$$

$$\begin{aligned}\Rightarrow\ \mathrm{TC}(\mathbb{R}P^n)\geq\inf\{k\in\mathbb{N}\mid\exists&\text{nowhere vanishing section of}\\&k(\xi\otimes\xi)\to\mathbb{R}P^n\times\mathbb{R}P^n\}=:A_n.\end{aligned}$$

Show that

$$A_n \geq \inf\{k \in \mathbb{N} \mid \exists \text{ non-singular } f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}\},$$

where $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ is non-singular if

(i) $f(\lambda u, \mu v) = \lambda\mu f(u, v) \ \ \forall \lambda, \mu \in \mathbb{R}, \ u, v \in \mathbb{R}^{n+1}$,

(ii) $f(u, v) = 0 \quad \Leftrightarrow \quad u = 0 \ \vee \ v = 0$.

Use topological result by Adem-Gitler-James '72 to connect this to immersions $\mathbb{R}P^n \to \mathbb{R}^k$ and derive the claim.

Remark (Gonzalez '05) There is a connection between TC and immersion dimension of lens spaces as well.

Symplectic manifolds (1)

Let (M, ω) be a $2n$-dimensional closed symplectic manifold, i.e. $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form.

Then $\sigma := 1 \times [\omega] - [\omega] \times 1 \in H^2(M \times M; \mathbb{R})$ satisfies

$$\sigma^{2n} = (-1)^n \binom{2n}{n} [\omega]^n \times [\omega]^n \neq 0,$$

so $\mathrm{cl}(Z_{\mathbb{R}}(M)) \geq 2n$ and thus

$$\mathrm{TC}(M) \geq 2n + 1.$$

Theorem

If M is simply connected, then $\mathrm{TC}(M) = 2n + 1$.

Proof.
By dimension/connectivity upper bound, $\mathrm{TC}(M) \leq \frac{2 \cdot 2n + 1}{2} + 1$, hence $\mathrm{TC}(M) \leq 2n + 1$. Combine with lower bound from above. $\square$

Symplectic manifolds (2)

(M, ω) closed symplectic manifold, $\dim M = 2n$.

Theorem (Grant-M., 2018)
If $f^*\omega = 0$ for all $f : T^2 \stackrel{C^0}{\to} M$, then $\mathrm{TC}(M) = 4n + 1$.

Method of proof Show that the condition on ω implies that $\mathrm{wgt}(\sigma) \geq 2$, use lower bound by weights,
$\mathrm{TC}(M) \geq 2n \cdot \mathrm{wgt}(\sigma) = 4n$.

Open problem What is $\mathrm{TC}(M)$ in general?

Personal guess Should be related to existence of symplectic torus actions.

Theorem (Grant, '12)
If X is a closed manifold that admits a non-trivial smooth semi-free T^k-action, then $\mathrm{TC}(X) \leq 2\dim X - k + 1$.

Collision-free motion planning problem

Simultaneous motion planning for several objects

Motion planning for k robots moving in the same workspace X corresponds to motion planning in X^k.

Problem Given starting points $x_1, \dots, x_k \in X$ and $y_1, \dots, y_k \in X$, find $\gamma_1, \dots, \gamma_k \in PX$ with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$ for all $i \in \{1, 2, \dots, k\}$.

Additionally, the robots should not collide during their movements.

Additional requirement $\gamma_1, \dots, \gamma_k$ should satisfy

$$\gamma_i(t) \neq \gamma_j(t) \quad \forall i \neq j,\ t \in [0, 1].$$

$\to$ Study configuration spaces of X.

Definition

Given a top. space X and $k \in \mathbb{N}$, $k \geq 2$, consider the *configuration space*

$$F(X,k) := \{(x_1,\dots,x_k) \in X^k \mid x_i \neq x_j \;\; \forall i \neq j\}.$$

To study the collision-free motion planning problem, compute $\mathrm{TC}(F(X,k))$ for suitable spaces X.

Theorem (Farber-Grant '09, Farber-Yuzvinsky '04)

Let $n,k \geq 2$. Then $\mathrm{TC}(F(\mathbb{R}^n,k)) = \begin{cases} 2k-1 & \text{if } n \text{ is odd,} \\ 2k-2 & \text{if } n \text{ is even.} \end{cases}$.

Problem This result is obtained using sophisticated techniques from algebraic topology. No explicit construction method for motion planners...

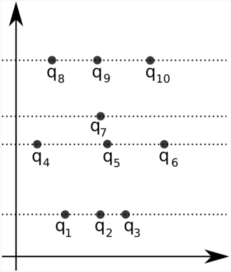
Construction of motion planners for $F(\mathbb{R}^2, k)$ (1)

(H. Mas-Ku, E. Torres-Giese, Motion planning algorithms for configuration spaces, 2015)

Define a partial order on $\mathbb{R}^2$ by

$$(x_1, y_1) \leq (x_2, y_2) \quad :\Leftrightarrow \quad y_1 < y_2 \ \vee \ (y_1 = y_2 \ \wedge \ x_1 \leq x_2.)$$

Let $q = (q_1, \ldots, q_k) \in F(\mathbb{R}^2, k)$ with $q_1 < q_2 < \cdots < q_k$.



Construction of motion planners for $F(\mathbb{R}^2, k)$ (2)

Then
$\exists r \in \mathbb{N},\ A(q) := (a_1, \ldots, a_r) \in \mathbb{N}^r$ with $|A(q)| := a_1 + \cdots + a_r = k$,

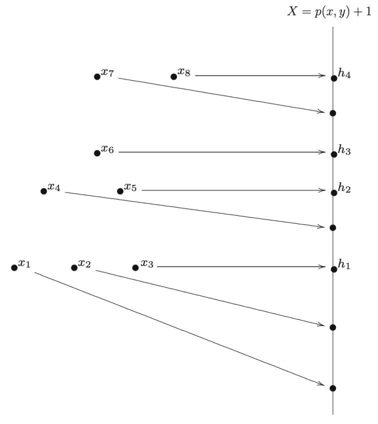
$$
\begin{aligned}
\pi_2(q_1) = \pi_2(q_2) = \cdots = \pi_2(q_{a_1}) &< \pi_2(q_{a_1+1}),\\
\pi_2(q_{a_1+1}) = \pi_2(q_{a_1+2}) = \cdots = \pi_2(q_{a_1+a_2}) &< \pi_2(q_{a_1+a_2+1})\\
&\vdots\\
\pi_2(q_{a_1+a_2+\cdots+a_{r-1}+1}) = \ldots &= \pi_2(q_k).
\end{aligned}
$$

For $A \in \bigcup_{r=1}^{k} \mathbb{N}^r$ with $|A| = k$ put
$F(A) := \{q = (q_1, \ldots, q_k) \in F(\mathbb{R}^2, k) \mid q_1 < \cdots < q_k,\ A(q) = A\}$.

Aim Given $A, B \in \bigcup_{r=1}^{k} \mathbb{N}^r$ with $|A| = |B| = k$, want to define motion planner $s_{A,B} : F(A) \times F(B) \to P(F(\mathbb{R}^2, k))$.

Construction of motion planners for $F(\mathbb{R}^2, k)$ (3)

Let $x = (x_1, \ldots, x_k) \in F(A)$, $y = (y_1, \ldots, y_k) \in F(B)$, put $p(x, y) := \max\{\max_i |\pi_1(x_i)|, \max_j |\pi_1(y_j)|\}$, where $\pi_1(a, b) = a$. Define $h \in F(\mathbb{R}^2, k) \cap (\{p(x, y) + 1\} \times \mathbb{R})$ by arranging points as in the picture (which is stolen from Mas-Ku, Torres-Giese)



Construction of motion planners for $F(\mathbb{R}^2, k)$ (4)

Define $h' \in F(\mathbb{R}^2, k) \cap (\{p(x, y) + 2\} \times \mathbb{R})$ in the same way with respect to y. Consider the paths $Q_x, \alpha_{x,y}, Q_y : [0, 1] \to F(\mathbb{R}^2, k)$ given by

- Q_x the straight line segment from x to h,
- Q_y the straight line segment from y to h',
- $\alpha_{x,y}$ - the straight line segment from h to h'.

With $*$ denoting concatenation, put

$$s_{A,B}(x, y) := Q_x * \alpha_{x,y} * \overline{Q}_y.$$

In this way, we obtain a continuous motion planner $s_{A,B} : F(A) \times F(B) \to P(F(\mathbb{R}^2, k))$.

Construction of motion planners for $F(\mathbb{R}^2, k)$ (5)

For $x = (x_1, \ldots, x_k) \in F(\mathbb{R}^2, k)$, $\exists! \sigma_x \in \Sigma_k$ with

$$x_{\sigma_x(1)} < x_{\sigma_x(2)} < \cdots < x_{\sigma_x(k)}.$$

For $\sigma \in \Sigma_k$ and A as above put

$$F(A, \sigma) := \{x = (x_1, \ldots, x_k) \in F(\mathbb{R}^2, k) \mid \sigma_x = \sigma, \\ (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \in F(A)\}.$$

Define $s_{A,B,\sigma,\tau} : F(A, \sigma) \times F(B, \tau) \to P(F(\mathbb{R}^2, k))$ by reordering the coordinates, applying $s_{A.B}$ and reordering the paths. Put

$$F_i := \bigcup_{(A,B) \in \mathbb{N}^i} \bigcup_{\sigma, \tau \in \Sigma_k} (F(A, \sigma) \times F(B, \tau)).$$

Construction of motion planners for $F(\mathbb{R}^2, k)$ (6)

One shows that

$$F(\mathbb{R}^2, k) \times F(\mathbb{R}^2, k) = \bigsqcup_{i=2}^{2k} F_i$$

and that the $s_{A.B,\sigma,\tau}$ assemble to form continuous motion planners

$$s_i : F_i \to P(F(\mathbb{R}^2, k)).$$

Thus, have explicit continuous motion planners over sets in a decomposition of $(F(R^2, k))^2$ with $2k - 1$ open sets.

$$(\Rightarrow \mathrm{TC}(F(\mathbb{R}^2, k)) \leq 2k - 1.)$$

This method was generalized by Farber ('17) to $F(\mathbb{R}^n, k)$.

Variations of topological complexity

- **Symmetric motion planning** (Farber-Grant '07)
  Consider only motion planners $s : X \times X \to PX$ with:
  - $s(x, x) =$ constant path for all $x \in X$,
  - $s(x, y) =$ inverse path of $s(y, x)$ for all $x, y \in X$.
- **Equivariant topological complexities** Different approaches to G-spaces and motion planners "behaving well" w.r.t. group actions.
  - (Colman-Grant '12) Equivariant TC
  - (Lubawski-Marzantowicz '14) Invariant TC
  - (Dranishnikov '15) Strongly equivariant TC
  - (Blaszczyk-Kaluba '18) Effective TC
- **Simplicial complexity** (Gonzalez '18)
  - combinatorial approach to motion planning in simplicial complexes
  - algorithmic methods to construct piecewise linear motion planners

Literature

For further introduction to topological complexity and more on topological robotics, see:

M. Farber, *Topology of robot motion planning*, in: Morse theoretic methods in nonlinear analysis and in symplectic topology, 185–230, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006.

M. Farber, *Invitation to topological robotics*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

Second part: Recent topics related to topological complexity

1. Topology and robot kinematics *(Pavesic, Murillo/Wu)*
2. Parametrized motion planning *(Cohen/Farber/Weinberger)*
3. Geodesic complexity *(Recio-Mitter)*
4. Spherical complexities *(M.)*

1. Topology and robot kinematics

Motivation

(after Pavešić '17, '18; Murillo-Wu '17)

Consider a *robot manipulator* with an *end effector*, i.e. some device/sensor/...., manipulating or measuring something.

In practice, we are only interested in the position of the end-effector. For example, if one uses a robot arm holding a screwdriver, on is often not interested in the position of the arm (i.e. the particular joints), but only in the position of the screwdriver.

Idea In practice, don't need motions between any two positions of the robot, we just need to get to each position of the end effector. Formalize this.

Forward kinematics

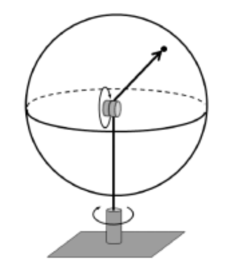
With each robot manipulator with end effector, we associate two spaces:

- $\mathcal{C}$ - the *configuration space*, the space of possible configurations/positions of the robot,
- $\mathcal{W}$ - the *work space*, the space of possible positions of the end effector.

The *forward kinematic map* or *work map* is a continuous map $f : \mathcal{C} \to \mathcal{W}$ associating with every configuration of the robot the corresponding position of the end effector.

Example for forward kinematics

Example A robot arm with one joint, rotating as indicated in the picture:



(stolen from P. Pavešić, A topologist's view of kinematic maps and manipulation complexity, many more examples in there)

Here, $\mathcal{C} = S^1 \times S^1$ and $\mathcal{W} = S^2$, so the forward kinematic map is $f : S^1 \times S^1 \to S^2$.

Inverse kinematics and topology

Would like a continuous map $s : \mathcal{W} \to \mathcal{C}$ associating with each position of the end effector a corresponding configuration, i.e. a continuous section of f.

Theorem (Gottlieb '86)

Let $n \in \mathbb{N}$, let $W = S^2$, $W = SO(3)$ or $W = SE(3)$. A continuous map $f : (S^1)^n \to W$ does not admit a continuous global section.

(Proof by standard algebraic topology).

Motion planning and the work map

Given an inital configuration $c \in \mathcal{C}$ and a desired end effector position $w \in \mathcal{W}$, we want to move the robot from c to a configuration $c' \in \mathcal{C}$ with $f(c') = w$.

Abstract problem Let C and W be path-connected top. spaces and $f : C \to W$ be continuous and surjective. Given $c \in C$ and $w \in W$, find $\gamma : [0,1] \overset{C^0}{\to} C$ with $\gamma(0) = c$ and $f(\gamma(1)) = w$.

Let $\pi : PC \to C \times C$, $\pi(\gamma) = (\gamma(0), \gamma(1))$, put $\pi_f := (\mathrm{id}_C \times f) \circ \pi : PC \to C \times W$. Want local sections of π_f.

Problem In general, π_f is not a fibration, so techniques for $\mathrm{secat}(\pi_f)$ might be non-available.

Approach of Murillo-Wu: (roughly) search for sections *up to homotopy.*

Topological complexity of a map

Definition (Pavešić '17)

Let C and W be path-connected top. spaces, $f : C \to W$ cont. and surjective, $\pi_f : PC \to C \times W$. Then $\mathrm{TC}(f) \in \mathbb{N} \cup \{+\infty\}$ is

$$\mathrm{TC}(f) := \inf\{n \in \mathbb{N} \mid \exists \emptyset = Q_0 \subset Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n = C \times W \\ \text{closed, s.t. } \forall j \, \exists s_j : Q_j \smallsetminus Q_{j-1} \overset{C^0}{\to} PC \text{ with } \pi_f \circ s_j = \mathrm{incl}_{Q_j \smallsetminus Q_{j-1}}\}.$$

Theorem If f is a fibration and W is a metrizable ANR, then $\mathrm{TC}(f) = \mathrm{secat}(\pi_f)$. In particular, if $C = W$, then $\mathrm{TC}(\mathrm{id}_C) = \mathrm{TC}(C)$.

Aim Compute $\mathrm{TC}(f)$ for examples occuring in actual problems from kinematics and find explicit motion planners with $\mathrm{TC}(f)$ domains of continuity.

In example from picture, Pavešić computed that $\mathrm{TC}(f) \leq 4$.

2. Parametrized motion planning

Motivation

(after Cohen-Farber-Weinberger ’20)

In several practical situations, one needs to do motion planning depending on parameters that might change occasionally, e.g.:

- a robot in a warehouse containing shelves, stacks of items or other obstacles, *whose position might change*,
- a submarine in the ocean with mines drifting and floating around.

In both cases, need to do motion planning in $X \setminus Q$ for Q a finite subset, but we want to keep our motion planners (as) "stable" (as possible) under small changes of the position of Q.

Collision-free motion planning with obstacles

Theorem (Fadell-Neuwirth '62)

Let X be a manifold (without boundary) with $\dim X \geq 2$. *Then*

$$p : F(X, m+n) \to F(X, m),$$
$$p(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) = (x_1, \ldots, x_m),$$

is a fiber bundle whose fiber is homeomorphic to

$$F(X \setminus Q_m, n),$$

where $Q_m \subset X$ is a set with m elements.

To do collision-free motion planning in X with m obstacles corresponds to motion planning "inside a fiber" of p.

$\longrightarrow$ want a notion of TC that is "stable under changing fibers"

Parametrized topological complexity

Definition (Cohen-Farber-Weinberger, September 2020)

Let $p : E \to B$ be a fibration with path-connected fiber and put $I := [0,1]$. Let $E \times_B E = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\}$ and put

$$E^I_B := \{\gamma \in C^0(I, E) \mid \exists b \in B \text{ s.t. } \gamma(I) \subset p^{-1}(\{b\})\}.$$

Then $\Pi : E^I_B \to E \times_B E$, $\Pi(\gamma) = (\gamma(0), \gamma(1))$, is a well-defined fibration and we put

$$\begin{aligned} \mathrm{TC}[p : E \to B] &:= \mathrm{secat}(\Pi : E^I_B \to E \times_B E) \\ &= \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = E \times_B E \text{ open cover, s.t. } \forall j\, \exists s_j : U_j \to E^I_B \\ &\qquad\qquad \text{with } \Pi \circ s_j = \mathrm{incl}_{U_j} \Big\}. \end{aligned}$$

Observations about parametrized TC

Let $p : E \to B$ be a fibration with path-connected fiber F.

- $\mathrm{TC}[p : E \to B] \geq \mathrm{TC}(F)$ and $\mathrm{TC}[p : E \to B] - \mathrm{TC}(F)$ can become arbitrarily big.
- If p is a principal G-bundle, then

$$\mathrm{TC}[p : E \to B] = \mathrm{TC}(G) = \mathrm{cat}(G).$$

- *An upper bound:* If $p : E \to B$ is a fiber bundle, E, B and F are CW complexes and F is r-connected, then

$$\mathrm{TC}[p : E \to B] \leq \frac{2\dim F + \dim B + 1}{r + 1} + 1.$$

- *A lower bound:* Let $\Delta_B : E \to E \times_B E$, $\Delta_B(e) = (e, e)$. Then

$$\mathrm{TC}[p : E \to B] \geq \mathrm{cl}\left(\ker\left[\Delta_B^* : H^*(E \times_B E; R) \to H^*(E; R)\right]\right) + 1.$$

Study the Fadell-Neuwirth fibrations $p : F(\mathbb{R}^k, m+n) \to F(\mathbb{R}^k, m)$, corresponding to collision-free motion planning for n robots in $\mathbb{R}^k$ with m mobile obstacles.

Theorem (Cohen-Farber-Weinberger '20)

a) *Let $k \geq 3$ be odd, $m \geq 2$ and $n \geq 1$. Then*

$$\mathrm{TC}[p : F(\mathbb{R}^k, m+n) \to F(\mathbb{R}^k, m)] = 2n + m.$$

b) *For $m, n \in \mathbb{N}$ it holds that*

$$\mathrm{TC}[p : F(\mathbb{R}^2, m+n) \to F(\mathbb{R}^2, m)] = 2n + m - 1.$$

Comparison with TC of the fiber

The fiber of $p : F(\mathbb{R}^k, m+n) \to F(\mathbb{R}^k, m)$ is $F(\mathbb{R}^k \setminus Q_m, n)$, where $Q_m \subset \mathbb{R}^k$ with $|Q_m| = m$.

Theorem (Farber-Grant-Yuzvinsky, '07)

Let $m \in \mathbb{N}$.

a) $\mathrm{TC}(F(\mathbb{R}^3 \setminus Q_m, n)) = 2n + 1$.

b) $\mathrm{TC}(F(\mathbb{R}^2 \setminus Q_m, n)) = \begin{cases} 2n & \text{if } m = 1, \\ 2n+1 & \text{if } m \geq 2. \end{cases}$

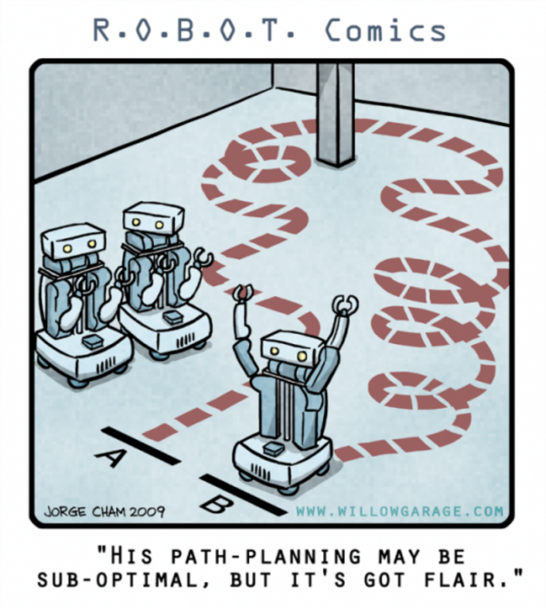
Thus,

$$\mathrm{TC}[p : F(\mathbb{R}^3, m+n) \to F(\mathbb{R}^3, m)] - \mathrm{TC}(F(\mathbb{R}^3 \setminus Q_m, n)) = m - 1$$

which becomes arbitrarily big for $m \to \infty$.

3. Geodesic complexity

The problem with topological motion planning



Motivation

(after David Recio-Mitter, '20)

Problem Motion planners with few domains of continuity may consist of paths that are not feasible or very inefficient.

Idea Make the additional requirement that all paths should have *minimal length*.

Shortest paths in metric spaces

Definition

Let (X, d) be a metric space and $\gamma : [0, 1] \to X$ be a rectifiable curve.

a) γ is called a *shortest path* if $\ell(\gamma) = d(\gamma(0), \gamma(1))$, where $\ell(\gamma) =$ length of γ.

b) γ is called a *minimal geodesic* if $\exists \lambda \geq 0$ with

$$d(\gamma(t_1), \gamma(t_2)) = \lambda |t_1 - t_2| \quad \forall t_1, t_2 \in [0, 1].$$

(minimal geodesic $\Rightarrow$ shortest path)

c) (X, d) is called *geodesic* if for any two $x, y \in X$ there exists a minimal geodesic from x to y.

Idea Study only motion planners $s : A \to PX$ for which $s(x, y)$ is a shortest path from x to y for all $(x, y) \in A$.

Definition (Recio-Mitter '20)

Let (X, d) a geodesic space, $GX := \{\text{minimal geodesics in } X\}$ and $\pi : GX \to X \times X$, $\pi(\gamma) = (\gamma(0), \gamma(1))$. The *geodesic complexity of X* is given by

$$\mathrm{GC}(X) := \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigsqcup_{j=1}^{n} A_j = X \times X, \text{ s.t. } A_j \text{ locally compact}, \\ \forall j\; \exists s_j : A_j \to GX \text{ with } \pi \circ s_j = \mathrm{incl}_{A_j} \Big\}.$$

Remark (1) (observed by Recio-Mitter) Would obtain the same number if we replaced GX by $\{\text{shortest paths in } X\}$.

(2) In Riemannian manifolds, get the usual definition of minimal geodesics.

(3) $\mathrm{TC}(X) \leq \mathrm{GC}(X)$ if X is an ENR (e.g. a Riemannian manifold).

Properties of geodesic complexity

Technical difficulty $\pi : GX \to X \times X$ is *not* a fibration, so can not use general results on secat etc.

Theorem $\mathrm{GC}(S^n) = \mathrm{TC}(S^n)$ for all $n \in \mathbb{N}$.

Proof Follows since the motion planners from part 1 are geodesic.

Theorem $\mathrm{GC}(\mathbb{R}P^n) = \mathrm{TC}(\mathbb{R}P^n)$ for all $n \in \mathbb{N}$.

Question Are there examples with $\mathrm{TC}(X) < \mathrm{GC}(X)$? If so, how big can $\mathrm{GC}(X) - \mathrm{TC}(X)$ become?

Geodesic complexity and the cut locus (1)

Definition

Let (X, d) be a metric space, $x \in X$. The *cut locus of* x is

$$C_x := \{y \in X \mid \exists \text{ more than one minimal geodesic from } x \text{ to } y\}.$$

The *total cut locus of* X is given by $C \subset X \times X$

$$C := \bigcup_{x \in X} (\{x\} \times C_x.)$$

Theorem (Blaszczyk/Carrasquel-Vera '18, Recio-Mitter '20)

There exists a continuous $s : (X \times X) \smallsetminus C \to GX$ *with* $\pi \circ s = \mathrm{incl}_{(X \times X) \smallsetminus C}$.

Proof Put $s(x, y) :=$ the unique minimal geodesic from x to y for all $(x, y) \notin C$. $\square$

Geodesic complexity and the cut locus (2)

$\Rightarrow$ Geodesic motion planning is completely determined by the total cut locus.

Problem The total cut locus can be *very* complicated, only very few explicit results.

Approach by Recio-Mitter Derive lower and upper bounds for $\mathrm{GC}(X)$ using stratifications of the total cut locus with "good properties".

Overview of results on geodesic complexity

Theorem (Recio-Mitter '20)

Consider T^2 with the flat Riemannian metric g_f and the metric g_e obtained from embedding T^2 in $\mathbb{R}^3$. Then

$$\mathrm{GC}(T^2, g_f) = \mathrm{TC}(T^2) = 3, \qquad \mathrm{GC}(T^2, g_e) = 4.$$

Theorem (Recio-Mitter '20)

a) *For every $k \in \mathbb{N}$ there exists a closed Riemannian manifold (M, g) with*

$$\mathrm{GC}(M, g) - \mathrm{TC}(M) \geq k$$

(Here, one might choose $M = S^{k+3}$ with a suitable metric.)

b) *For each $k \in \mathbb{N}$ there exists a metric g_k on $\mathbb{R}^{k+1}$ with*

$$\mathrm{GC}(\mathbb{R}^{k+1}, g_k) \geq k + 1 \quad \textit{(while } \mathrm{TC}(\mathbb{R}^{k+1}) = 1\textit{)}.$$

4. Spherical complexities

A warning

Warning: shameless self-promotion!

after:

M., Spherical complexities with applications to closed geodesics, to appear in Algebr. Geom. Topol., 2019.

M., Existence results for closed Finsler geodesics via spherical complexities, Calc. Var., 2020.

Estimating numbers of critical points

Theorem (Lusternik-Schnirelmann '34, Palais '65)

Let M be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on M. Then

$$\#\mathrm{Crit}\, f \geq \mathrm{cat}(M).$$

Method of proof Study $\mathrm{cat}_M(f^{-1}((-\infty, a]))$ for increasing a, use minimax methods.

Problem When studying G-invariant functions on Hilbert manifolds, where G a Lie group, one wants to count *G-orbits of critical points* à la Lusternik-Schnirelmann.

Helpful approach Study equivariant versions of LS-category, approach of Clapp-Puppe, Bartsch, ...

Example: the closed geodesics problem

Let M be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric (e.g. $F(x, v) = \sqrt{g_x(v, v)}$ for g Riemannian metric), $\Lambda M := H^1(S^1, M) = W^{1,2}(S^1, M)(\simeq C^0(S^1, M))$,

$$E_F : \Lambda M \to \mathbb{R}, \quad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))^2 \, dt.$$

Then ΛM is a Hilbert manifold, E_F is $C^{1,1}$ and S^1-invariant and satisfies the Palais-Smale condition (Mercuri, '77).

$$\text{Crit } E_F = \{\text{closed geodesics of } F\} \cup \{\text{constant loops}\}.$$

Two problems for L-S approach:

1. Critical points occur in S^1-orbits.
2. $\{$constant loops in $M\}$ is a critical submanifold of ΛM.

How to disregard these phenomena?

Definition of spherical complexities (M., 2019)

Let X top. space, $n \in \mathbb{N}_0$, $B_{n+1}X := C^0(B^{n+1}, X)$,
$S_nX := \{f \in C^0(S^n, X) \mid f \text{ is nullhomotopic}\}$.

Definition

Let $A \subset S_nX$. A *sphere filling on A* is a continuous map $s : A \to B_{n+1}X$ with $s(\gamma)|_{S^n} = \gamma$ for all $\gamma \in A$.

$$\mathrm{SC}_{n,X}(A) := \inf\Big\{ r \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{r} U_j \supset A \text{ open cover, sphere fillings } s_j : U_j \to B_{n+1}X \ \forall j \in \{1, 2, \dots, r\} \Big\} \in \mathbb{N} \cup \{\infty\}.$$

Call $\mathrm{SC}_n(X) := \mathrm{SC}_{n,X}(S_nX)$ the *n-spherical complexity of X*.

Remark $\mathrm{SC}_0(X) = \mathrm{TC}(X)$, the topological complexity of X.

Properties of spherical complexities

Let X be a metrizable ANR (e.g. a locally finite CW complex).

Proposition Let $c_n : X \to S_nX$, $(c_n(x))(p) = x$ for all $p \in S^n$, $x \in X$. Then $\mathrm{SC}_{n,X}(c_n(X)) = 1$.

Consider the left $O(n+1)$-actions on S_nX and $B_{n+1}X$ by reparametrization, i.e.

$$(A \cdot \gamma)(p) = \gamma(A^{-1}p) \qquad \forall \gamma \in S_nX,\ A \in O(n+1),\ p \in S^n.$$

Proposition Let $G \subset O(n+1)$ be a closed subgroup and $\gamma \in S_nX$ and let G_γ denote its isotropy group. If G_γ is trivial or $n = 1$, then $\mathrm{SC}_{n,X}(G \cdot \gamma) = 1$.

Proof for G_γ trivial: Take $\beta \in B_{n+1}X$ with $\beta|_{S^n} = \gamma$, put

$$s : G \cdot \gamma \to B_{n+1}X, \quad s(A \cdot \gamma) = A \cdot \beta\ \forall A \in G.$$

A Lusternik-Schnirelmann-type theorem for SC_n

Theorem (M., 2019)

Let $G \subset O(n+1)$ be a closed subgroup, $\mathcal{M} \subset S_n X$ be a G-invariant Hilbert manifold, $f \in C^{1,1}(\mathcal{M})$ be G-invariant. Put $f^a := f^{-1}((-\infty, a])$. Let

$$\nu(f, a) := \#\{\text{non-constant } G\text{-orbits in } \operatorname{Crit} f \cap f^a\}.$$

If

- *f satisfies the Palais-Smale condition w.r.t. a complete Finsler metric on $\mathcal{M}$,*
- *f is constant on $c_n(X)$,*
- *G acts freely on $\operatorname{Crit} f \cap f^a$ or $n = 1$,*

then $$\mathrm{SC}_{n,X}(f^a) \leq \nu(f, a) + 1.$$

Main theorem for closed geodesics

Let M be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric and let $E_F : H^1(S^1, M) \cap S_1M \to \mathbb{R}$ be its energy functional.

Theorem

Let $\nu(F, a)$ be the number of $SO(2)$-orbits of non-constant contractible closed geodesics of F of energy $\leq a$. Then

$$\nu(F, a) \geq \mathrm{SC}_{1,M}(E_F^a) - 1.$$

If F is reversible, the same holds for the number of $O(2)$-orbits of contractible closed geodesics.

Remark The counting does not distinguish iterates of the same prime closed geodesic.

Results on closed geodesics

Theorem (Lusterník-Fet, '51, for Riemannian manifolds)

Every Finsler metric on a closed manifold admits a non-constant closed geodesic.

Definition Two closed geodesics $\gamma_1, \gamma_2 : S^1 \to X$ are *geometrically distinct* if $\gamma_1(S^1) \neq \gamma_2(S^1)$. They are called *positively distinct* if they are either geom. distinct or $\exists A \in O(2) \setminus SO(2)$ with $\gamma_1 = A \cdot \gamma_2$.

- Bangert-Long, 2007: every Finsler metric on S^2 has two positively distinct ones
- Rademacher, 2009: every *bumpy* Finsler metric on S^n has two positively distinct ones
- etc., Long-Duan 2009 for S^3, Wang 2019 for pinched metrics on S^n, ...

New results using spherical complexities

Theorem (M., 2020)

Let M be a closed oriented manifold, $F : TM \to [0, +\infty)$ be a Finsler metric of reversibility λ and flag curvature K. Let $\ell_F > 0$ be the length of the shortest non-const. closed geodesic of F.

a) *If $M = S^{2d}$, $d \geq 2$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda}\sqrt{g_1}$, then F admits two pos. distinct closed geodesics of length $< 2\ell_F$. (g_1 = round metric of constant curvature 1)*
b) *If $M = S^{2d+1}$, $d \in \mathbb{N}$, $\frac{\lambda^2}{(1+\lambda)^2} < K \leq 1$ and $F \leq \frac{(k+1)(1+\lambda)}{m\lambda}\sqrt{g_1}$, then F admits $\lceil \frac{2m}{k} \rceil$ pos. distinct closed geodesics of length $< (k+1)\ell_F$.*
c) *If $M = \mathbb{C}P^n$ or $M = \mathbb{H}P^n$, $n \geq 3$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda}\sqrt{g_1}$, then $\exists$ two pos. distinct closed geodesics of length $< 2\ell_F$.*

Literature on the second part (1)

1. Topology and robot kinematics

    - P. Pavešić, *A topologist's view of kinematic maps and manipulation complexity*, in: Topological complexity and related topics, 61–83, Contemp. Math., 702, Amer. Math. Soc., Providence, RI, 2018.
    - P. Pavešić, *Complexity of the forward kinematic map*, Mechanism and Machine Theory 117 (2017), 230-243.
    - A. Murillo and J. Wu, *Topological complexity of the work map*, to appear in J. Topol. Anal., arXiv:1706.09157.
    - D. Gottlieb, *Robots and fibre bundles*, Bull. Soc. Math. Belg. Sér. A 38 (1986), 219–223.

Literature on the second part (2)

2. Parametrized motion planning
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Thank you for your attention!

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