CLASSIFICATION OF VECTOR BUNDLES OVER MANIFOLDS

A HOT STEW W/ A BIT OF HOMOTOPY THEORY, DIFFERENTIAL GEOMETRY AND ALGEBRAIC TOPOLOGY.

MOTIVATION: vector bundles tell us a lot about a manifold, e.g.

1. Does your manifold admit a certain geometric structure? (e.g. is it almost complex? is it almost contact?)
2. Can a manifold be embedded in a certain $\mathbb{R}^N$? (the normal bundle must have certain properties → does any vector bundle over the manifold fullfill these properties?).
3. the sphere bundles of a v.b. are usually interesting examples of manifolds.

TRY TO UNDERSTAND WHICH V.B. OVER A FIXED MANIFOLD EXIST BY MEANS OF COMPUTABLE INVARIANTS.

A WAY TO CLASSIFY V.B.

Let M be a compact, connected, oriented smooth manifold.

most of the following is also true for more general spaces!

$\pi: E \to M$ a real v.b. over M of rank k

- Since M is compact we have that E is a subbundle of a trivial bundle $M \times \mathbb{R}^N$ (N big enough) i.e. $\exists f: E \to \mathbb{R}^N$ injective linear map on the fibers, i.e. $f|_{E_p}: E_p \to \mathbb{R}^N$ is a monomorphism for all $p \in M$.

  $E_p := \pi^{-1}(p)$

- this defines a map $M \longrightarrow Gr_k(\mathbb{R}^N)$ k-dimensional linear subspaces of $\mathbb{R}^N$ "Grassmanian"

  $$M \ni p \longmapsto f(E_p) \leq \mathbb{R}^N$$

- eliminate the "unknown" numbers N by taking "limits"

  $$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \dots \subset \mathbb{R}^N \subset \dots \subset \mathbb{R}^\infty = \bigcup_{N \in \mathbb{N}} \mathbb{R}^N$$

$$\longrightarrow \quad Gr_k(\mathbb{R}^N) \subset Gr_k(\mathbb{R}^{N+1}) \subset \dots \subset Gr_k(\mathbb{R}^\infty) =: BO(k)$$

- THM: Isomorphism classes of real k-dim v.b's over M correspond bijectively to homotopy classes of maps $M \longrightarrow Gr_k(\mathbb{R}^\infty) = BO(k)$

i.e. $\boxed{Vect_k(M) \xleftrightarrow{1:1} [M, BO(k)]}$

homotopy classes of maps $M \to BO(k)$

Isomorphism classes of k-dim v.b. over M

- Same is true for complex v.b. $\quad Vect_k^{\mathbb{C}}(M) \cong [M, BU(k)]$

oriented v.b $\quad Vect_k^{+}(M) \cong [M, BSO(k)]$

- $\exists$ k-dim v.b. $\hat{\pi} : \hat{E} \longrightarrow BO(k)$ s.t the bijection $Vect_k(M) \cong [M, BO(k)]$ is given by pullback

universal bundle

$$[M, BO(k)] \longrightarrow Vect_k(M)$$
$$[g] \longmapsto [g^* \hat{E}]$$

homotopic map pull back $\hat{E}$ to isomorphic v.b.'s

EXAMPLE: $M = S^{n+1}$ $\quad \mathrm{Vect}^+_k(S^{n+1}) = [S^{n+1}, BSO(k)] = \pi_{n+1} BSO(k)$.

$\pi_{n+1} BSO(k) \cong \pi_n SO(k)$, for $k > n+1$ $\pi_n SO(k)$ is independent of k! And the homotopy groups are periodic (Bott periodicity)

$$\pi_{8\ell+0} SO(k) = \mathbb{Z}_2$$
$$\pi_{8\ell+1} SO(k) = \mathbb{Z}_2$$
$$\pi_{8\ell+2} SO(k) = 0$$
$$\pi_{8\ell+3} SO(k) = \mathbb{Z}$$
$$\pi_{8\ell+4} SO(k) = 0$$
$$\pi_{8\ell+5} SO(k) = 0$$
$$\pi_{8\ell+6} SO(k) = 0$$
$$\pi_{8\ell+7} SO(k) = \mathbb{Z}$$

COROLLARY: if $S^{8\ell+5}$ is embedded into a manifold M of dimension bigger than $16\ell+10$, then the normal bundle of $S^{8\ell+5}$ in M is trivial!

$\rightsquigarrow$ some interesting consequenc. (see maybe later)

$$\dim \nu = \dim M - (8n+5)$$
$$> 16\ell+10-8\ell-5 = 8\ell+5$$
$$= \dim S^{8\ell+5}$$

$(n = 8\ell+4)$

For complex v.b. the situation is a bit easier:

$\pi_{n+1} BU(k) \cong \pi_n U(k)$, for $2k > n$ the groups are independent of k! (Bott Periodicity)

$$\pi_{2l}\, U(k) = 0 \;;\; \pi_{2l+1}\, U(k) = \mathbb{Z} \qquad (l = 0, 1, \dots)$$

EXAMPLE: Complex line bundles: one computes that

$$Gr_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty \qquad \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \cdots \subset \mathbb{CP}^N \subset \cdots \subset \mathbb{CP}^\infty$$

and that $\mathbb{CP}^\infty$ is an $K(\mathbb{Z}, 2)$, i.e.

$Vect_1^{\mathbb{C}}(M) \cong [M, \mathbb{CP}^\infty] \cong H^2(M; \mathbb{Z})$ and the isomorphism is given by

$$Vect_1^{\mathbb{C}}(M) \ni L \longmapsto c_1(L) \in H^2(M; \mathbb{Z})$$

(real line bundles $Gr_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$; $Vect_1(M) = [M, \underset{K(\mathbb{Z}_2,1)}{\mathbb{RP}^\infty}] = H^1(M; \mathbb{Z}_2)$

iso is given by $Vect_1(M) \ni L \mapsto w_1(L) \in H^1(M; \mathbb{Z}_2)$)

IN GENERAL it is difficult to compute $[M, B(\ldots)]$.

BUT in low dimensions still possible!

- There are a lot of results on classification of v.b.'s in low dimensions. We will consider a certain result which is for this talk interesting.

Thm (Woodward 1981) Let X be a connected CW-complex of dimension n, where $n = 4, 6$, let

$$\alpha : [X, BSO(n)] \longrightarrow H^2(X; \mathbb{Z}_2) \oplus H^4(X; \mathbb{Z}) \oplus H^n(X; \mathbb{Z})$$

defined by

$$\alpha(E) := (w_2(E), p_1(E), e(E))$$

Then: (a)

$$\operatorname{im}(\alpha) = \begin{cases} \{(x,y,z) : \wp_4(y+2z) = \mathfrak{P}x\} & n=4 \\ \{(x,y,z) : \wp_2(y) = x^2,\ \wp_2(z) = 0\} & n=6 \end{cases}$$

(b) α is injective if $H^4(X;\mathbb{Z})$ & $H^n(X;\mathbb{Z})$ have no 2-torsion
(e.g if X is a 4-manifold, or if X is a 6-manifold with vanishing odd integer cohomology)

(c) if $x \in H^2(X;\mathbb{Z}_2)$ then $x = w_2(E)$ for some $E \in [X, BSO(n)]$
iff $2\beta_4 \mathfrak{P} x = 0$

<u>REMARK</u>: For the cases $n = 3,7$ there are similar results.

But the case $n=5$ is missing. Why?

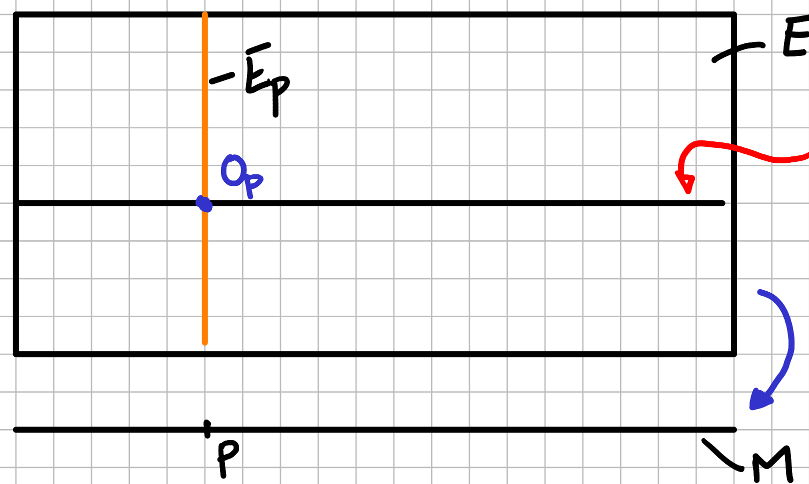
Because e.g. on S^5 the tangent bundle of S^5 is <u>not</u> trivial, but has the <u>same</u> characteristic classes as the trivial bundle over S^5!

$$\mathrm{Vect}_5^+(S^5) = \pi_4\, SO(5) \cong \pi_4\, Sp(2) \cong \mathbb{Z}_2$$

Side Dish: Characteristic Classes

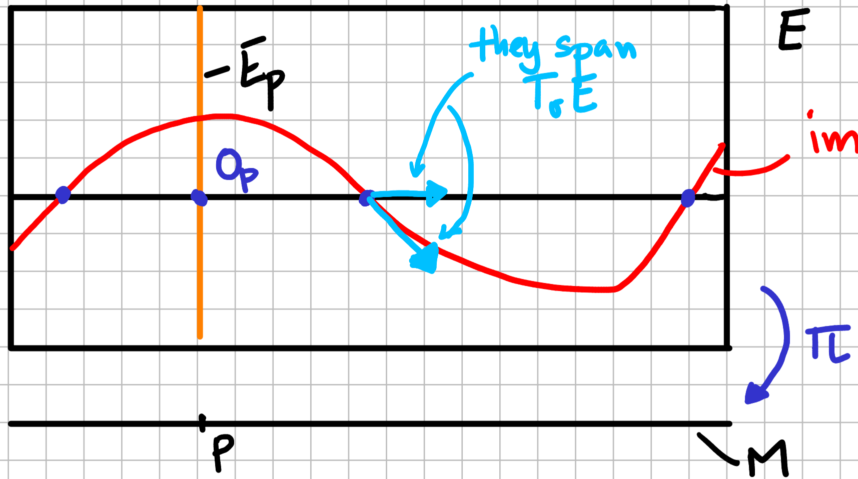
The Euler Class

- $\pi: E \longrightarrow M$ oriented v.b. & M oriented manifold.
- the image of the zero section of $\pi: E \to M$ is naturally a submanifold of E diffeomorphic to M. (We will write $M \subset E$ for this submanifold)



- Since $M \subset E$ is a submanifold, it is possible to deform any section $s: M \to E$ to intersect $M \subset E$ transversally.

[transversal: $\forall\, v \in \mathrm{im}(s) \cap M \subset E$: $T_v M + T_v \mathrm{im}(s) = T_v E$]



⚠ If two submanifolds intersect each other transversally then their intersection is again a submanifold!

The zero locus of s $Z(s)$; i.e where $\mathrm{im}(s)$ intersects $M \subset E$ is a submanifold of dimension

$$\boxed{\dim M - \mathrm{rk}(E)}$$

- The Poincaré dual of $Z(s)$ is a cohomology class in $H^{\mathrm{rk}(E)}(M;\mathbb{Z})$ and is defined as the EULER-CLASS $e(E)$.
- Why EULER? The number $\langle e(TM), [M] \rangle$ (kronecker pairing; $[M]$ = fundamental class of M.) is equal to the EULER CHARACTERISTIC of M, ie. $\sum_{i=0}^{\dim M} (-1)^i \dim H^i(M;\mathbb{R})$.

$H^i(M) \cong H_{n-i}(M)$

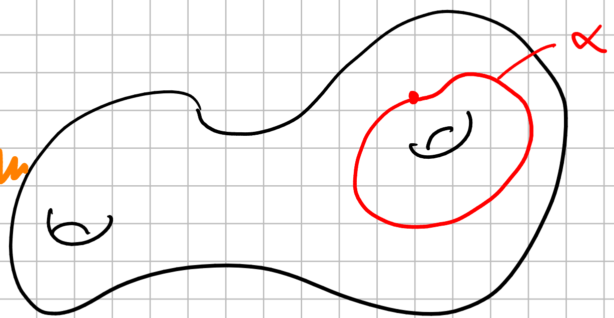
- COROLLARY: If E admits a nowhere vanishing section then $e(E)=0$. (if $\mathrm{rk}(E) = \dim M$ then $e(E)=0$ implies existence of nowhere vanishing section)

SPIN STRUCTURES

(presented as exotic soup)

- Stiefel-Whitney classes (SW classes) are cohomology classes $w_i \in H^i(M;\mathbb{Z}_2)$ associated to a v.b. $\pi: E \to M$.
- Let us have a look on their geometric interpretation:

$w_1(E) \in H^1(M;\mathbb{Z}_2)$: E is orientable iff for every loop $\alpha: S^1 \to M$ the orientation of the fibers is preserved as one goes around the loop.



This gives a homomorphism

$\pi_1(M) \to \mathbb{Z}_2$

which is 0 if the orientation is preserved and 1 else.

Since $\mathbb{Z}_2$ is abelian the map $\pi_1(M) \to \mathbb{Z}_2$ factors trough the abelianization of $\pi_1(M)$ giving a homomorphism

$$H_1(M) \to \mathbb{Z}_2$$

i.e. it represents an element of

$$H^1(M; \mathbb{Z}_2)$$

which we call the 1st SW-class of E $w_1(E)$.

NOTE: Over S^1 there are only two real v.b.s since $\pi_1 BO(n) \cong \pi_0 O(n) = \mathbb{Z}_2$. They are distinguished by $w_1(E)$! $\longrightarrow$ $w_1(E)$ indicates if E is trivial or not over the 1-skeleton of M!
(i.e. E is orientable if it is trivial over the 1-skeleton!)

$$\boxed{\text{NOW ASSUME THAT } E \to M \text{ is an oriented v.b. over } M}$$

FRAMINGS OF V.Bs OVER S^1:

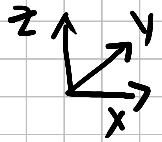
If $E \to S^1$ is an oriented v.b over S^1 than it is trivial. A choice of a trivialization is a set of sections $s_1, \dots, s_k : S^1 \longrightarrow E$ $(k = rk(E))$ s.t. $(s_1(p), \dots, s_k(p))$ is a basis of E_p $\forall p \in S^1$.

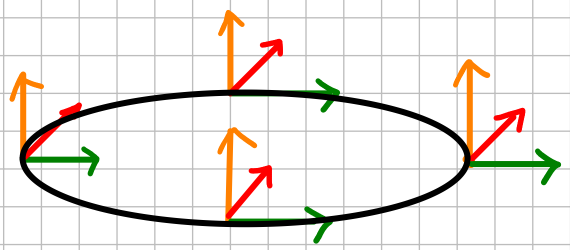
A framing is a homotopy class of a trivialization, such that the homotopy is given by trivializations.

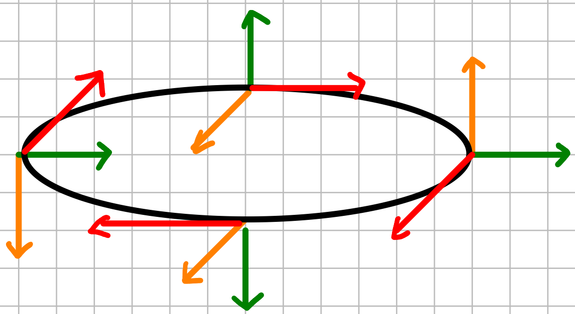
There are exactly two framings for $E \longrightarrow S^1$ for $rk\, E \geq 3$. If you choose a trivialization, then any other trivialization is given by a map $S^1 \longrightarrow SO(k)$. This defines an element of $\pi_1 SO(k) \cong \mathbb{Z}_2$.

$S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$

framings of $T\mathbb{R}^3|_{S^1}$



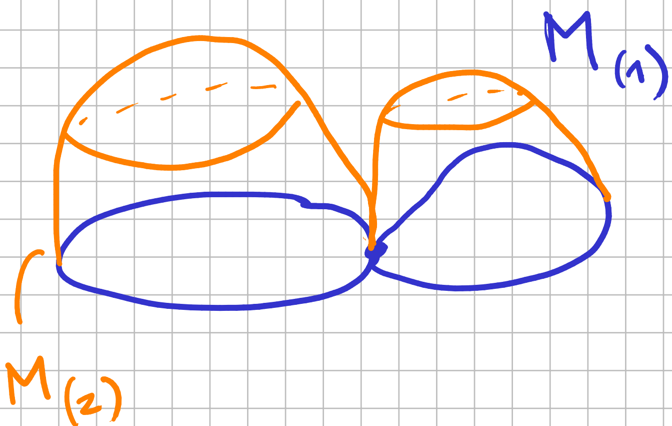




SPIN STRUCTURES:

$E \to M$ oriented v.b.

- Choose a framing of E over the 1-skeleton of your manifold.
- If this framing can be extended over the 2-skeleton of M then we say that E has a spin structure.



obstruction lives in $H^2(M;\mathbb{Z}_2)$

BACK TO DIMENSION 5

We're interested in classification of oriented rank 5 v.b. over compact oriented and connected 5-manifolds M

This is equivalent to say that $w_1(E)=0$ & $w_2(E)=0$

Thm (Čadek & Vanžura 1993)

S^5 does not fullfil this condition

if M does not posses any spin structure (i.e. $w_2(M) \neq 0$)
then

$$\gamma : [M, BSO(5)] \longrightarrow H^2(M;\mathbb{Z}_2) \oplus H^4(M;\mathbb{Z}_2) \oplus H^4(M;\mathbb{Z})$$
$$E \longmapsto (w_2(E), w_4(E), p_1(E))$$

has the properties

(i) $\operatorname{im}(\gamma) = \{ (a,b,c) : \rho_4 c = \mathfrak{P} a + i_* b \}$

(ii) γ is injective if $H^4(M;\mathbb{Z})$ has no element of order 2.