BACK TO DIMENSION 5

we're interested in classification of oriented rank 5 v.b. over compact oriented and connected 5-manifolds M

This is equivalent to say that $w_1(E)=0$ & $w_2(E)=0$

Thm (Čadek & Vanžura 1993)

S^5 does not fullfil this condition

if M does not posses any spin structure (i.e. $w_2(M)\neq 0$) then

$$\gamma: [M, BSO(5)] \longrightarrow H^2(M;\mathbb{Z}_2)\oplus H^4(M;\mathbb{Z}_2)\oplus H^4(M;\mathbb{Z})$$
$$E \longmapsto (w_2(E), w_4(E), p_1(E))$$

has the properties

(i) $\operatorname{im}(\gamma) = \{(a,b,c) : \rho_4 c = \mathfrak{P} a + i_* b\}$

(ii) γ is injective if $H^4(M;\mathbb{Z})$ has no element of order 2.

Case $w_2(M)=0$ (M has a spin structure)

Let us consider the following set of v.b.

$$W := \{ V \in [M, BSO(5)] : w_2(V) = w_4(V) = 0 \}$$

v.bs over M w/ spin structure & w_4 equal zero.

Thm (K., '21)

W is has a group structure and is isomorphic to

$$\boxed{2H^4(M;\mathbb{Z}) \oplus \mathbb{Z}_2}$$

But How?

$$\boxed{\begin{aligned} Vect_5^+(S^5) &= \pi_5 BSO(5) = \pi_4 SO(5) \\ &= \pi_4 Sp(2) = \mathbb{Z}_2 \end{aligned}}$$

- BLACKBOX : $W \approx [M, BSU(2)]$ complex rk 2 v.b.

$\operatorname{rk} V = 5 \quad w_2(V) = w_4(V) = 0$

then $V \cong E \oplus \underline{\mathbb{R}}$ & E

$$[M, BSU(2)] = \{E \in [M, BU(2)] : \det E \text{ is trivial}\}$$

$\Updownarrow$

$\exists$ complex volume form

$\Updownarrow$

$c_1(E) = 0$

- $\underline{\underline{\text{But}}}$ $BSU(2) = BSp(1) = \mathbb{HP}^\infty$

$S^4 = \mathbb{HP}^1 \subset \mathbb{HP}^2 \subset \dots \subset \mathbb{HP}^n \subset \dots$

$\mathbb{HP}^\infty = \varinjlim \mathbb{HP}^k$

CW structure

$\mathbb{HP}^\infty = S^4 \cup e^8 \cup e^{12} \cup \dots$

elementary homotopy theory gives us then:

$$[M, BSU(2)] = [M, S^4]$$

since M is of dimension 5 !

$$[M, S^4] = \pi^4(M)$$ 4^{th} cohomotopy group of M.

$\dim M = n \qquad \pi^k(M)$ is a group if $k < 2n-1$

COMPUTATION OF $\pi^4(M)$ 　　 ($\pi^n(M^{n+1})$ 　 $n \geq 4$)

$S^4 \hookrightarrow K(\mathbb{Z},4)$

we have an obvious map

$$\pi^4(M) \longrightarrow H^4(M;\mathbb{Z})$$
$$[f] \longmapsto f^*(\sigma) \ ; \ \sigma \in H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$$

or dually

$$\pi^4(M) \longrightarrow H_1(M)$$
$$[f] \longmapsto [f^{-1}(x_0)] \ ; \ x_0 \in S^4 \text{ regular value}$$

- This map is a homomorphism which is surjective :

  choose an element $[C] \in H_1(M)$; it can be represented by an embedded circle $C \subset M$.

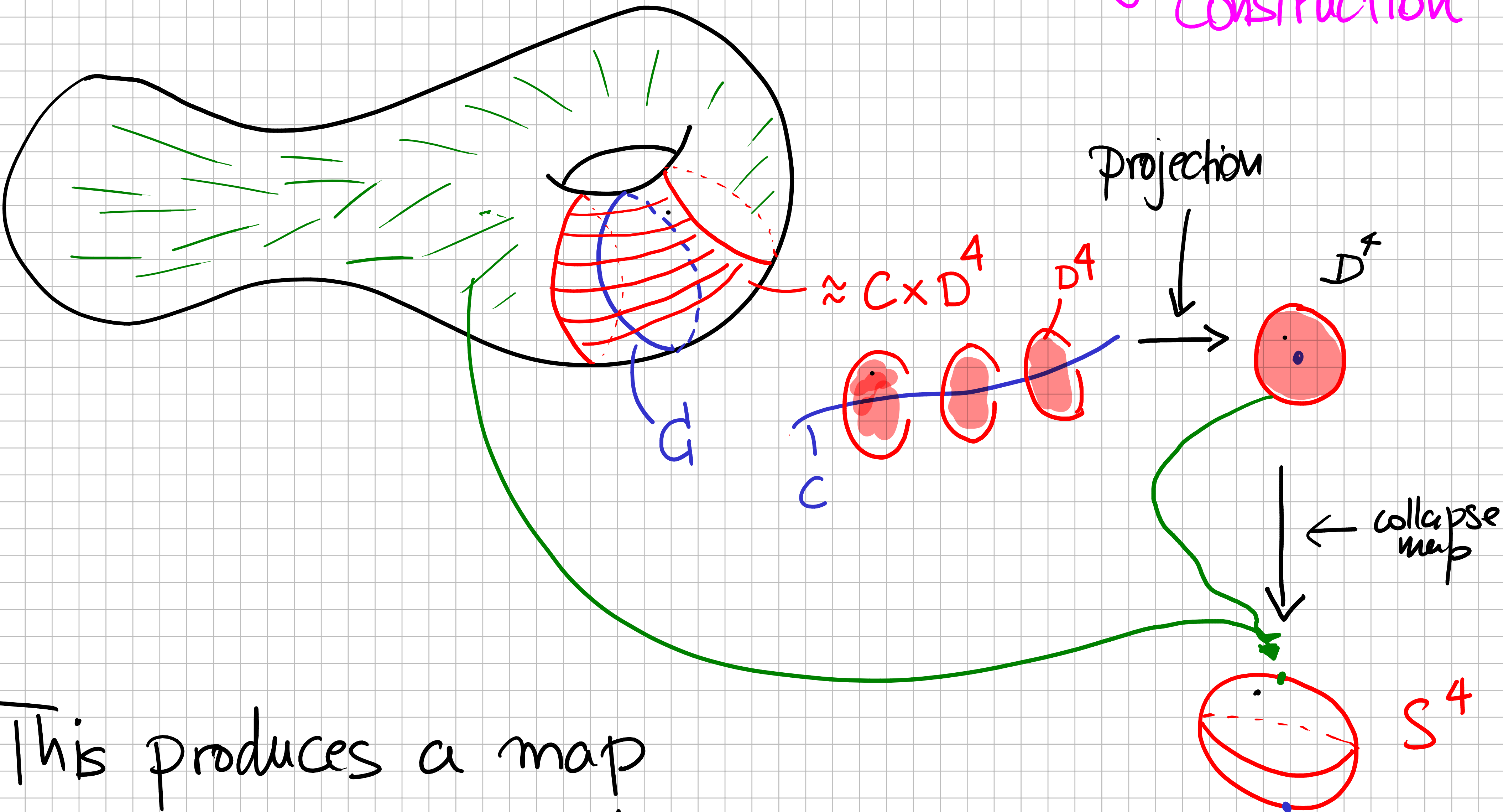
  we have seen that the normal bundle must be trivial (since C & M are oriented).

  Choose a framing of the normal bundle

i.e.

$$\nu(C \hookrightarrow M) \cong C \times \mathbb{R}^4$$

Pontryagin-Thom Construction



This produces a map

$$f : M \longrightarrow S^4$$

Pontryagin–Thom construction gives a bijection

$$\pi^4(M) \overset{1:1}{\longleftrightarrow} \Omega_1^{fr}(M)$$

$$\Omega_1^{fr}(M) = \left\{ (C, \varphi_v) : \begin{array}{l} C \text{ compact 1-dim. subm. of } M \ \& \\ \varphi_v \text{ is a framing of } \nu(C) \end{array} \right\} \Big/ \sim$$

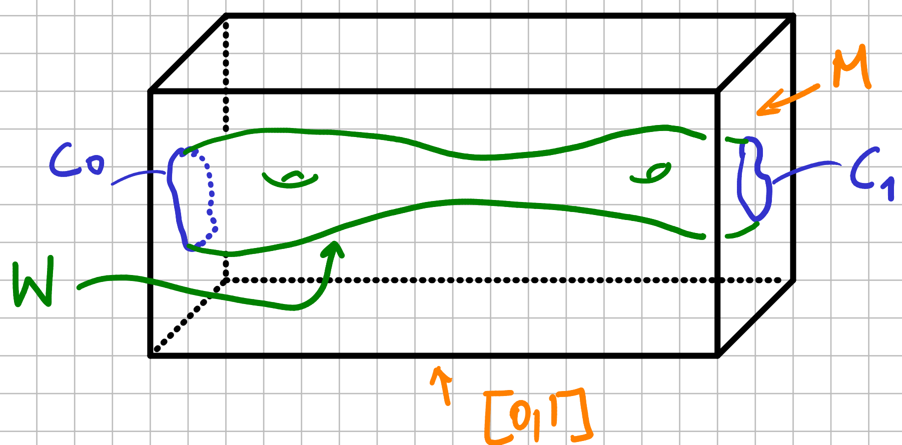
$(C_0, \varphi_0) \sim (C_1, \varphi_1) \;:\Leftrightarrow\; \exists\, W^2 \subset M \times [0,1]$ s.t. $\underbrace{W \cap M \times \{0\}}_{C_0} \cup \underbrace{W \times M \times \{1\}}_{C_1} = \partial W$

$\&\ \exists$ framing φ of $\nu(W^2)$

s.t. $\varphi\big|_{C_0} = \varphi_0 \qquad \varphi\big|_{C_1} = \varphi_1$

"normally framed bordism between (C_0,φ_0) & (C_1,φ_1)"

$M \times [0,1]$



- We have so far

$$0 \to \text{Kernel} \longrightarrow \pi^4(M) \longrightarrow H_1(M) \longrightarrow 0$$

can only be 0 or $\mathbb{Z}_2$!

- We show that Kernel $= \mathbb{Z}_2$ by constructing a non-trivial homomorphism $K : \pi^4(M) \to \mathbb{Z}_2$ s.t. $K|_{\text{kernel}}$ is an isomorphism.

$\Rightarrow$ K is a splitting map for the s.e.s hence

$$\pi^4(M) \cong \text{Kernel} \oplus H_1(M)$$

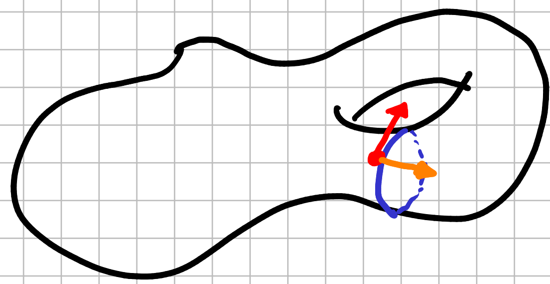
SIDE DISH:

$\pi^4(M) = [M, BSU(2)]$; $S^4 = \mathbb{H}P^1$ $\quad H \to \mathbb{H}P^1$ tautological bundle.

$f : M \to S^4$ $\quad E = f^*(H)$. $f^{-1}(x_0) = Z(s)$ $s \in \Gamma(H)$ P.D to $e(E)$.

K will assign the stable framing of $(C, \varphi_\gamma) = [f]$

- We have



$$\underline{\mathbb{R}}^5 \cong TM|_C \cong TC \oplus \gamma(c) \overset{\varphi_\gamma}{\cong} TC \oplus \underline{\mathbb{R}}^4$$

since M has a spin structure

This gives a stable framing of C and thus can be identified w/ $\Omega_1^{fr}(S^5) \cong \pi^4(S^5) = \pi_5 S^4 \cong \mathbb{Z}_2$

Pontryagin-Thom

1st stable hom. group

- **Prop.:** The map

$$K: \pi^4(M) \longrightarrow \Omega_1^{fr}(S^5) \cong \mathbb{Z}_2$$
$$[f] = [C, \varphi_\nu] \longmapsto [C, \varphi_s]$$

stable framing

is a homomorphism. Furthermore $K|_{\text{kernel}}$ is an iso.

$\Longrightarrow$ short exact sequence splits, thus

$$\boxed{\pi^4(M) \cong H_1(M) \oplus \Omega_1^{fr}}$$

FINALLY

Since $W = \{ V \in [M, BSO(5)] : w_2(V) = 0 = w_4(V) \}$

The condition $w_4(V) = 0$ implies $e(E) \equiv 0 \bmod 2$ for $V = E \oplus \underline{\mathbb{R}}$.

$V \cong E \oplus \underline{\mathbb{R}}$

Thus

$$\boxed{W \cong 2H^4(M; \mathbb{Z}) \oplus \mathbb{Z}_2}$$

$$V \longmapsto (e(E), k(E))$$

rank n ov. S^n $\pi_{n-1} SO(n)$

FURTHER PROPERTIES OF K

Thm (K. '21)

Let $E \to B$ be a rank n v.b. over M^{n+1} w/ $w_2(E)=0$.

Then:

E has a nowhere vanishing section if

$e(E)=0$ & $K(E)=0$

THANK YOU &

ENJOY YOUR MEAL !